# RECENT RESEARCH DEVELOPMENTS IN

# Physics

Vol. 5 - 2004 Part III

MANAGING EDITOR : S.G. PANDALAI

TRANSWORLD RESEARCH NETWORK

Recent Research Developments in

# Physics

Published by Transworld Research Network

2004; Rights Reserved
Transworld Research Network
T.C. 37/661(2), Fort P.O.,
Trivandrum-695 023, Kerala, India

**Managing Editor**
S.G. Pandalai

**Publications Manager**
A. Gayathri

Transworld Research Network assumes no responsibility for the opinions and statements advanced by contributors

ISBN: 81-7895-126-6

# Contents

TRANSWORLD RESEARCH NETWORK

Recent Research Developments in

# Physics

Vol. 5 (2004) Part III
Managing Editor: S.G. Pandalai

Transworld Research Network
37/661 (2), Fort P.O., Trivandrum-695 023, Kerala, India

Recent Res. Devel. Physics, 5(2004): 1487-1510 ISBN: 81-7895-126-6

# 55 Duality and canonical transformations in the scalar field theory

**Chueng-Ryong Ji, Yuriy Mishchenko and Abouzeid Shalaby**
Department of Physics, North Carolina State University, Raleigh, NC 27695-8202, USA

## Abstract

*We discuss a self-organizing nature of quantum fields using explicit examples of the low dimensional scalar field theories. We utilize a nonperturbative approach known as the Oscillator Representation (OR) method which appears to be much simpler than other well-known nonperturbative methods. The key idea of the OR method is to make canonical transformations between the original particle theory and the quasiparticle theory and find non-trivial vacuum solutions which satisfy the self-consistency conditions required by a desired form of the quasiparticle effective Hamiltonian. A link between the OR method and the Effective Action approach can also be made with a specic canonical transformation of field shifts. In the low-dimensional scalar field theories, we find the duality property between the original particle theory and the quasiparticle theory, i.e. the original nonperturbative strong interaction theory is equivalent to the weakly interacting*

Correspondence/Reprint request: Dr. Chueng-Ryong Ji, Department of Physics, North Carolina State University, Raleigh NC 27695-8202, USA. E-mail: ji@ncsu.edu

*quasiparticle theory. We present explicit examples of the duality which allows the conversion of the original nonperturbative strong interaction problem into a weekly interacting quasiparticle problem that can be solved by the usual perturbation technique. The renormalization procedure is also described for the different levels of loop calculations such as involving one-loop, normal-ordering and two-loop regularizations. Various numerical results of nontrivial vacuum solutions, including their energy densities, quasiparticle mass spectra and classical effective potentials, are discussed along with their symmetry properties.*

# I. Introduction

With the current advances of Relativistic Heavy Ion Collision (RHIC) physics, there is a growing interest in discussing self-organizing nature of relativistic quantum field theory (RQFT). The phase transition and the spontaneous symmetry breaking anticipated to be observed in the RHIC facilities are the paramount examples of the physical phenomena due to the self-organization of quantum fields. Since these novel phenomena cannot be easily described in the ordinary perturbation series, they form highly nontrivial examples that ought to be analyzed by nonperturbative methods available for the RQFTs. Although the problem of quantization of interacting fields is not yet completely solved, duality existing in the RQFTs may be the key to handle the strong interacting theories. As the interactions get stronger, the associate quantum fluctuations grow and consequently the formation of a new non-trivial vacuum, often accompanied by the condensation of fields, may be energetically favorable. The particles moving in the non-trivial vacuum acquire mass from the interaction with the condensates and may be described by quasiparticles with different mass relative to the original particle mass. Then, the coupling of the quasiparticles may get weaker as the coupling of the original theory gets stronger and thus the original particle theory with strong coupling can be equivalent (*i.e.* dual) to a quasi-particle theory with weak coupling. When such duality works self-consistently within the RQFT, one may be able to convert the original nonperturbative problem into the quasiparticle perturbative problem and utilize the perturbation theory of quasiparticles in the strong coupling regime of the original theory.

In this work, we discuss self-organizing nature of RQFTs in the explicit examples of low-dimensional scalar field models. This involves the aspects of symmetry breaking and restoration due to the quantum fluctuations which have been discussed in the past [1-3]. To analyze the self-organizing properties, we utilize a nonperturbative approach that appears to be much simpler than other rather well-known methods such as the Hartree Approximation (HA) and the Gaussian Effective Potential (GEP). Following the nomenclature in the literature, we call this method as the Oscillator Representation (OR) method, although in our view all of these nonperturbative methods (OR,GEP,HA) stem from the same principle of quantum Effective Action[4].

The OR method was explicitly formulated by Efimov[5] and was, in fact, used implicitely earlier by Chang [6] and Magruder [7]. The method is at length described in a monograph [8]. The basic idea of the OR method is to redefine the mass of interacting field relative to the free one and simultaneously introduce a shift of the field quantization point leading to a nonzero value of its vacuum condensate. This effect is realized in the nature by a spontaneous symmetry breaking mechanism and thus one can expect that it may be a general feature of quantum field systems with strong interactions. To our knowledge, the OR method has not been utilized as extensively as the methods of GEP[9-15] and HA[16, 17]. Recently, the OR method was used to investigate the phase structure of $(\phi^4 + \phi^6)_{1+1}$ theory[18]. In this work, we make an extension to the case of $(\phi^4 + \phi^6)_{2+1}$ theory.

Formally, the OR is settled by requiring that the Hamiltonian operator $H = H_0 + H_I$ should be written in terms of creation and annihilation operators of an oscillator basis with an appropriate frequency and in the correct form defined by[8]:
(1) the Hamiltonian $H_0$ is quadratic in the field operators,
(2) the interaction Hamiltonian $H_I$ contains field operators only in powers greater than two. Considering these requirements with the canonical transformation of the field variables, one may introduce a properly defined representation of the original theory in terms of quasiparticles. Such representations correspond to possible phases of the theory.

As we will discuss later, for given values of coupling constants, the exact OR results may include several vacuum solutions with different mass values for the quasiparticle fields. Among those multiple solutions, however, it is certainly not difficult to select the non-trivial solutions that are physically meaningful because of the duality considerations. In this work, we will focus on those duality-related solutions which are not hindered from the physical interpretation.

The structure of this paper is as follows. In section II, we illustrate the formulation of the OR method for a simple example of the $\phi^4$ scalar field theory. The relation between the OR method and the effective action formulation is also described in this section. Section III is devoted to the application of the OR method for the investigation of the phase structure of the $(\phi^4+\phi^6)_{2+1}$ theory. We present first the calculations up to one-loop and then extend the calculation to include the non-cactus diagrams up to two-loops. The results are discussed in Section IV. We find two physically meaningful duality-related quasiparticle solutions with different symmetry properties and show that they are, indeed, equivalent descriptions of the original theory in the strong coupling regime of the original theory. Summary and conclusion follows in SectionV.

# II. Formulation of the OR method

## A. The OR method for the $\phi^4$ theory

Let us illustrate the OR method in the example of $\phi^4$ scalar field theory. The Hamiltonian density for the $\phi^4$ theory is given by

$$H = \frac{1}{2}\left((\nabla\phi)^2 + \pi^2 + m_0^2\phi^2\right) + \frac{g_0}{4!}\phi^4, \tag{1}$$

where $m_0$ and $g_0$ are the bare mass and coupling, respectively, and $\pi = \dot{\phi}$ is the canonical conjugate momentum. We follow the systematic renormalization procedure described in Ref.[19]. The renormalization starts by introducing the renormalized field as $\phi = \sqrt{Z}\phi_r$, where $Z$ is the field renormalization constant. This leads to the Hamiltonian in the form

$$H = \frac{Z}{2}\left((\nabla\phi_r)^2 + m_0^2\phi_r^2\right) + \frac{1}{2Z}\pi_r^2 + \frac{Z^2 g_0}{4!}\phi_r^4, \tag{2}$$

where $\pi = \frac{1}{\sqrt{Z}}\pi_r$. Note that each bare parameter in Eq.(1) can be rewritten as a sum of a renormalized finite parameter and a counter term. This means that the Hamiltonian could be written as

$$H = H_0 + H_I + H_{ct}, \tag{3}$$

where

$$\begin{aligned} H_0 &= \frac{1}{2}\left(\pi^2 + (\nabla\phi)^2 + m^2(\mu)\phi^2\right), \\ H_I &= \frac{g(\mu)}{4!}\phi^4, \\ H_{ct} &= \frac{1}{2}\left(\frac{1}{Z} - 1\right)\pi^2 + \frac{1}{2}\delta Z\,(\nabla\phi)^2 + \delta m^2(\mu)\phi^2 + \frac{\delta g(\mu)}{4!}\phi^2. \end{aligned} \tag{4}$$

Here, the subscript $r$ is dropped and instead the parameter $\mu$ is introduced as a sliding renormalization scale. Because of the equivalence between the two Hamiltonian forms, Eqs.(2) and (3), the following relations should be fulfilled:

$$\begin{aligned} Z(\mu)m_0^2 &= m^2(\mu) + \delta m^2(\mu), \quad Z^2(\mu)g_o = g(\mu) + \delta g(\mu), \\ Z(\mu)h_0 &= h(\mu) + \delta h(\mu) \;\text{ and }\; \delta Z(\mu) = Z(\mu) - 1. \end{aligned} \tag{5}$$

The quantization procedure can be further introduced by defining the field operators as

$$\begin{aligned} \phi(x) &= \int \frac{d\vec{k}}{2\pi\sqrt{2w}} \left\{a(\vec{k})\exp\left(i\vec{k}.\vec{x} - iwx_o\right) + a^\dagger(\vec{k})\exp\left(-i\vec{k}.\vec{x} + iwx_o\right)\right\}, \\ \pi(x) &= \frac{1}{i}\int \frac{d\vec{k}}{2\pi}\sqrt{\frac{w}{2}} \left\{a(\vec{k})\exp\left(i\vec{k}.\vec{x} - iwx_o\right) - a^\dagger(\vec{k})\exp\left(-i\vec{k}.\vec{x} + iwx_o\right)\right\}, \end{aligned} \tag{6}$$

where the energy-momentum dispersion relation is given by $w(k) = \sqrt{\vec{k}^2 + m^2(\mu)}$ and the particle creation $a^\dagger(\vec{k})$ and annihilation $a(\vec{k})$ operators satisfy the commutatior relation $[a(\vec{k}), a^\dagger(\vec{k}')] = \delta(\vec{k} - \vec{k}')$. Also, the vacuum is defined by

$$a(\vec{k})\,|0\rangle \text{ and } \quad \langle 0\,|0\rangle = 1.$$

In Eq.(3), the renormalization scheme (R-scheme) should be fixed, *e.g.* an $R_\mu$ clas is chosen in such a way that the ratio $\frac{m(\mu)}{\mu}$ is constant [8]. This is because the chang of the ratio $\frac{m(\mu)}{\mu}$ would change the R-scheme[20]. The representation in Eq.(6) is meaningful only if the couplings are small enough to permit the free field approximation. When the couplings become strong, quantum corrections induced by $H_I$ become large and the representation in terms of the free fields (Eq.(6)) is no longer valid. To avoid this problem it is instructive to consider the canonical transformation of the original field representation that replaces the original strongly interacting theory with an equivalent effective theory with quasiparticle degrees of freedom.

In this work, we consider the canonical transformation of the field shift followed by the general Bogoliubov transformation [8]:

$$\alpha(\vec{K}) = \cosh\xi(\vec{K})a(\vec{K}) - \sinh\xi(\vec{K})a^{\dagger}(-\vec{K}), \tag{7}$$

If parameter $\xi$ is chosen in the form

$$\xi(\vec{K}) = \frac{1}{2}\ln\left\{\varsigma^2\frac{\omega(\vec{K})}{\Omega(\vec{K})}\right\}, \tag{8}$$

where $\zeta$ is a constant and

$$\Omega(\vec{K}) = \sqrt{(\vec{K})^2 + M^2}, \tag{9}$$

then the transformation in Eq.(7) induces the scale transformation of the field and the mass of the field changes from $m$ to $M$. In other words, projecting the transformation into the fields leads to

$$(\phi,\pi) \to \left(\zeta\psi, \frac{1}{\zeta}\Pi\right). \tag{10}$$

Note that, if $\zeta = 1$, *i.e.* there is no scale transformation, then Eq. (7) becomes

$$a_M(k) = \frac{1}{2}\left(\sqrt{\frac{w_m(k)}{w_M(k)}} + \sqrt{\frac{w_M(k)}{w_m(k)}}\right)a_m(k) - \frac{1}{2}\left(\sqrt{\frac{w_m(k)}{w_M(k)}} - \sqrt{\frac{w_M(k)}{w_m(k)}}\right)a_m^{\dagger}(-k), \tag{11}$$

$$a_M^{\dagger}(k) = \frac{1}{2}\left(\sqrt{\frac{w_m(k)}{w_M(k)}} + \sqrt{\frac{w_M(k)}{w_m(k)}}\right)a_m^{\dagger}(k) - \frac{1}{2}\left(\sqrt{\frac{w_m(k)}{w_M(k)}} - \sqrt{\frac{w_M(k)}{w_m(k)}}\right)a_m(-k), \tag{12}$$

where $w_m(k) = \sqrt{\vec{k}^2 + m^2}$ and $\omega_M(k) = \sqrt{\vec{k}^2 + M^2}$. This transformation shows how the free field operators with mass $M$ can be expressed in terms of the field operators with mass $m$. Also, this transformation is canonical and thus satisfies the same commutation relation as before,

$$\left[a_M(k), a_M^\dagger(k')\right] = \delta(k-k'). \tag{13}$$

Now, to get the quasiparticle effective Hamiltonian, we first apply the canonical transformation of the field shift followed by scale and mass change[8], *i.e.*

$$(\phi, \pi)\underset{m \longrightarrow M}{\longrightarrow}\left(\varsigma(\psi + B), \frac{1}{\varsigma}\Pi\right), \tag{14}$$

where we explicitly separate an allowed constant field condensate as $B$. Also, the field $\psi$ has mass $M = t{\cdot}m$. For the R-scheme to be equivalent in the two representations the canonical transformation should be accompanied by the scale transformation $\mu \to \nu = t\mu$ [8]. After applying these transformations, the Hamiltonian in Eq.(3) can be written in the form

$$H = \bar{H}_0 + \bar{H}_I + \bar{H}_{ct} + \bar{H}_1 + E, \tag{15}$$

where

$$\bar{H}_0 = \frac{1}{2}\left(\Pi^2 + (\nabla\psi)^2\right) + \frac{1}{2}M^2\psi^2, \tag{16}$$

$$\begin{aligned}
\bar{H}_I &= \frac{g(\nu)}{4!}\left(\psi^4 + 4B\psi^3\right), \\
\bar{H}_{ct} &= \frac{1}{2}\left(\left(\frac{1}{\bar{Z}} - 1\right)\Pi^2 + (\bar{Z} - 1)(\nabla\psi)^2\right) + \frac{1}{2}\left(\delta m^2(\nu) + \frac{\delta g(\nu)}{2}B^2\right)\psi^2 \\
&+ \frac{\delta g(\nu)}{4!}\left(\psi^4 + 4B\psi^3\right) + \left(\delta m^2(\nu) + \frac{\delta g(\nu)}{3!}B^2\right)B\psi + \frac{1}{2}\delta m^2(\nu)B^2 + \frac{\delta g(\nu)}{4!}B^4,
\end{aligned}$$

with $\bar{Z} = \varsigma^2 Z$, also

$$\bar{H}_1 = \frac{1}{2}\left(m^2(\nu) - M^2 + \frac{g(\nu)}{2}B^2\right)\psi^2 + \left(m^2(\nu) + \frac{g(\nu)}{3!}B^2\right)B\psi, \tag{17}$$

and

$$E = \frac{1}{2}m^2(\nu)B^2 + \frac{g(\nu)}{4!}B^4. \tag{18}$$

According to the OR method requiring a correct form of the resulting Hamiltonian, each term in $\bar{H}_1$ should vanish identically. Thus we have

$$0 = \left( m^2(\nu) + \frac{g(\nu)}{3!} B^2 \right) B, \tag{19}$$

$$0 = \left( m^2(\nu) - M^2 + \frac{g(\nu)}{2} B^2 \right). \tag{20}$$

Since Eq.(19) leads to the Klein-Gordon equation for the constant field $B$, $B$ is certainly a classical solution for the field $\phi$. In other words, the quantum fluctuations are built on top of the classical background field $B$. Also, in Eqs.(19) and (20), the parameters $m(\nu)$ and $g(\nu)$ are calculated using loop expansions. In 1+1 dimensions, the one-loop calculation is equivalent to the normal-ordering[21]. In this case, the OR equations describe a trivial ($b = 0$) solution and two non-trivial solutions ($b \neq 0$) which develop above a certain critical coupling[8, 18]. The order parameter $b$ in these solutions is discontinuous and thus OR gives first order phase transition. This situation persists even when perturbative corrections, up to $g^3$, are included in the energy density of the broken-symmetry (BS) phase in Eq.(18). Also, application of OR method in $(\phi^4)_{1+1}$ theory does not give the correct phase structure[22]. This is due to the highly nonperturbative character of the theory at the critical region. The situation can be improved by using resummation of the perturbation series[23].

For the $(\phi^4)_{2+1}$ theory, up to two-loops, the OR method gives a non-trivial but symmetric (S) phase. The phase structure of the $(\phi^4 + \phi^6)_{2+1}$ in the OR method has not been investigated yet and thus we present it in section III.

## B. The relation between the OR method and the Effective Action formulation

To show the field theoretic background of the OR equations in the case of the transformation $\phi \rightarrow \psi + B$, consider the energy functional defined by[19]

$$Z[J] = \exp(-iE[J]) = \int D\phi \exp\left( i \int d^4x \, (L[\phi] + J\phi) \right), \tag{21}$$

where $L[\phi]$ is the Lagrangian density, $J$ is an external source and $E[J]$ is the energy functional. The functional derivative of $E[J]$ with respect to $J(x)$ yields the result

$$\frac{\delta}{\delta J(x)} E[J] = \frac{-\int D\phi \exp\left( i \int d^4x \, (L[\phi] + J\phi) \right) \phi(x)}{\int D\phi \exp\left( i \int d^4x \, (L[\phi] + J\phi) \right)} = -\langle M | \phi | M \rangle_J, \tag{22}$$

where $<M|\phi|M>_J$ is the vacuum expectation value of the field $\phi$, *i.e.* the classical field B. The Effective Action is defined by the Legendre transformation

$$\Gamma(B) = -E[J] - \int d^4x J(x)B(x). \tag{23}$$

Taking the functional derivative of the Effective Action $\Gamma$ with respect to $B(x)$ (by analogy with the derivative of the free energy with respect to the magnetization in statistical physics), one can easily obtain

$$\frac{\delta}{\delta B(x)}\Gamma[B] = \frac{\delta}{\delta B(x)}E[J] - \int d^4x \frac{\delta J(x)}{\delta B(x)}B(x) - J(x) = -J(x). \tag{24}$$

If $B(x)$ is translationally invariant (as in the case of constant $B$), we can write $\Gamma[B]$ in the form

$$\Gamma[B] = -(VT)V_{eff}(B), \tag{25}$$

where $V_{eff}(B)$ is the Effective Potential. If the external source $J(x)$ is zero, then it equals to the vacuum energy density. In this case, the functional derivative is simplified to the partial derivative and Eq.(24) can be read as

$$\frac{\partial V_{eff}(B)}{\partial B} = 0. \tag{26}$$

Using the energy density in Eq.(18), Eq.(26) for $V_{eff}(B)$ results in

$$\frac{\partial E}{\partial B} = \left(m^2(\nu) + \frac{g(\nu)}{3!}B^2\right)B = 0, \tag{27}$$

which is exactly the first OR equation given by Eq.(19). By calculating the second functional derivative, we obtain the result presented in Ref.[19]:

$$\frac{\delta^2}{\delta B(x)\delta B(y)}E[B] = -iD^{-1}(x-y), \tag{28}$$

where $D$ is the Feynman propagator. When $B$ is invariant under translation, this simplifies as

$$\frac{\partial^2}{\partial B^2}E[B] = -iD^{-1}(0) = M^2, \tag{29}$$

which again, using the energy density equation(Eq.(18)), results in

$$\frac{\partial^2}{\partial B^2}E = \left(m^2(\nu) + \frac{g(\nu)}{2}B^2\right) = M^2. \tag{30}$$

This is exactly the second OR equation given by Eq.(20). In general, we have

$$\frac{\delta^n}{\delta B(x_1)\delta B(x_2)\ldots\ldots\ldots\delta B(x_n)}\Gamma[B] = -i\left\langle \phi(x_1)\ldots\ldots\ldots\phi(x_n)\right\rangle_{1PI}, \tag{31}$$

where $\langle\phi(x_1)\ldots\ldots\ldots\phi(x_n)\rangle_{1PI}$ is the one-particle-irreducible n-point function. This relation gives the coupling constant renormalization condition:

$$\frac{\partial^4 E}{\partial B^4} = g(\nu). \tag{32}$$

Thus, the OR method with the canonical transformation of field shift is linked to the Effective Action formulation. The OR method can also use other canonical transformations, although we do not know if the Effective Action can exist for other canonical transformations. In this sense, the OR method may be regarded as more general than the Effective Action approach.

# III. Application of the OR method to the $(\phi^4 + \phi^6)_{2+1}$ theory

Let us now consider the application of OR to a more complicated theory of $(\phi^4 + \phi^6)_{2+1}$. Proceeding the same way as before, the Hamiltonian density for the $(\phi^4 + \phi^6)$ theory is given by

$$H = \frac{1}{2}\left((\nabla\phi)^2 + \pi^2 + m_0^2\phi^2\right) + \frac{g_0}{4!}\phi^4 + \frac{h_0}{6!}\phi^6, \tag{33}$$

which can be rewritten as

$$H = H_0 + H_I + H_{ct}, \tag{34}$$

where

$$\begin{aligned} H_0 &= \frac{1}{2}\left(\pi^2 + (\nabla\phi)^2 + m^2(\mu)\phi^2\right), \\ H_I &= \frac{g(\mu)}{4!}\phi^4 + \frac{h(\mu)}{6!}\phi^6, \\ H_{ct} &= \frac{1}{2}\left(\frac{1}{Z} - 1\right)\pi^2 + \frac{1}{2}\delta Z(\nabla\phi)^2 + \delta m^2(\mu)\phi^2 + \frac{\delta g(\mu)}{4!}\phi^2 + \frac{\delta h(\mu)}{6!}\phi^6. \end{aligned} \tag{35}$$

We apply the canonical transformation of the field shift and simultaneously apply the scale transformation $\mu \to \nu$. This results in

$$H = \bar{H}_0 + \bar{H}_I + \bar{H}_{ct} + \bar{H}_1 + E, \tag{36}$$

where the different terms are given by

$$\bar{H}_0 = \frac{1}{2}\left(\Pi^2 + (\nabla\psi)^2\right) + \frac{1}{2}M^2\psi^2, \tag{37}$$

$$\begin{aligned}
\bar{H}_I &= \frac{g(\nu)}{4!}\left(\psi^4 + 4B\psi^3\right) + \frac{h(\nu)}{6!}\left(\psi^6 + 6B\psi^5 + 15B^2\psi^4 + 20B^3\psi^3\right), \\
\bar{H}_{ct} &= \frac{1}{2}\left(\left(\frac{1}{\bar{Z}} - 1\right)\Pi^2 + \left(\bar{Z} - 1\right)(\nabla\psi)^2\right) + \frac{1}{2}\left(\delta m^2(\nu) + \frac{\delta g(\nu)}{2}B^2 + \frac{\delta h(\nu)}{4!}B^4\right)\psi^2 \\
&+ \frac{\delta h(\nu)}{6!}\left(\psi^6 + 6B\psi^5 + 15B^2\psi^4 + 20b^3\psi^3\right) + \frac{\delta g(\nu)}{4!}\left(\psi^4 + 4B\psi^3\right) \\
&+ \left(\delta m^2(\nu) + \frac{\delta g(\nu)}{3!}B^2 + \frac{\delta h(\nu)}{5!}B^4\right)B\psi \\
&+ \frac{1}{2}\delta m^2(\nu)B^2 + \frac{\delta g(\nu)}{4!}B^4 + \frac{\delta h(\nu)}{6!}B^6.
\end{aligned} \tag{38}$$

Here, the residual term $\bar{H}_1$ can be found as

$$\bar{H}_1 = \frac{1}{2}\left(m^2(\nu) - M^2 + \frac{g(\nu)}{2}B^2 + \frac{h(\nu)}{4!}B^4\right)\psi^2 + \left(m^2(\nu) + \frac{g(\nu)}{3!}B^2 + \frac{h(\nu)}{5!}B^4\right)B\psi, \tag{39}$$

and the energy density of the phase is given by

$$E = \frac{1}{2}m^2(\nu)B^2 + \frac{g(\nu)}{4!}B^4 + \frac{h(\nu)}{6!}B^6. \tag{40}$$

The correct form of the Hamiltonian demands that each term in $\bar{H}_1$ vanishes identically, that is

$$0 = \left(m^2(\nu) + \frac{g(\nu)}{3!}B^2 + \frac{h(\nu)}{5!}B^4\right)B, \tag{41}$$

$$0 = \left(m^2(\nu) - M^2 + \frac{g(\nu)}{2}B^2 + \frac{h(\nu)}{4!}B^4\right). \tag{42}$$

These are the OR equations for the $(\phi^4 + \phi^6)_{2+1}$ scalar theory. Their solutions describe possible effective representations of the original theory in terms of quasiparticles with mass $M$. The dependence of the parameters $m(\nu)$, $g(\nu)$ and $h(\nu)$ on the sliding scale $\nu$ is calculated using loop expansions as shown below.

## A. One-loop calculations

The $(\phi^4 + \phi^6)_{2+1}$ theory is renormalizable, i.e. there are a finite number of divergent amplitudes, but the divergence exists in each order of the perturbation series. Up to

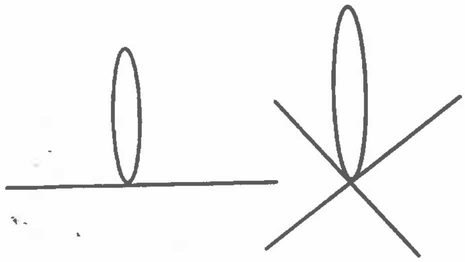

**Figure 1.** The one-loop divergent diagrams of the $(\phi^4 + \phi^6)_{(2+1)}$.

one-loop, the only divergent diagrams are shown in Fig.1. In the first diagram of Fig.1, the four-momentum of the initial particle is of couse same as that of the final particle and we denote it as $p$. In the second diagram of Fig.1, the four-momenta of the initial (final) particles are denoted by $p_1$ and $p_2$ ($p_3$ and $p_4$), respectively.

With this in mind, the self-energy amplitude $M(p^2)$ and the two-particle scattering amplitude have contributions both from the renormalized Hamiltonian $H = \bar{H}_0 + \bar{H}_1$ and the counter terms. To calculate the counter terms we use the following renormalization conditions[19]:

1. The propagator is set equal to $\frac{i}{p^2 - m^2}$ in the limit $p^2 \to m^2$, specifying the location of the pole and it's residue.
2. The two-particle scattering amplitude is set equal to $-ig$ at s=$(p_1 + p_2)^2 = 4m^2$, t= $(p_1 - p_3)^2$ = u = $(p_1 - p_4)^2 = 0$, where s, t and u are the Mandelstam variables.
3. For the six-point function, there are no divergent diagrams contributing at this level so that the relevant counter term is zero.

Both of the diagrams in Fig.1 will contribute to their corresponding amplitudes ($M(p^2)$ and $-ig$) with the following terms in 2+1 dimensions, respectively,

$$\frac{(-ig)}{2} \int \frac{d^3k}{(2\pi)^3} \frac{i}{k^2 - m^2}, \tag{43}$$

$$\frac{(-ih)}{2} \int \frac{d^3k}{(2\pi)^3} \frac{i}{k^2 - m^2}. \tag{44}$$

Using the dimensional regularization and applying the renormalization conditions, we get the following counter terms:

$$\delta m^2(\nu) = -\frac{g(\nu)}{2(4\pi)^{\frac{d}{2}}} \frac{\Gamma(1-\frac{d}{2})}{(m^2(\nu))^{1-\frac{d}{2}}},$$

$$\delta g(\nu) = -\frac{h(\nu)}{2(4\pi)^{\frac{d}{2}}} \frac{\Gamma(1-\frac{d}{2})}{(m^2(\nu))^{1-\frac{d}{2}}},$$
$$\delta h(\nu) = 0, \tag{45}$$

where $d$ is the space-time dimension and $\Gamma(x)$ is the Euler Gamma function. Replacing the parameters $\frac{g}{4!} \to \tilde{g}$ and $\frac{h}{6!} \to \tilde{h}$, the counter terms take the form (as $d \to 3$)

$$\delta m^2(\nu) = \frac{12\tilde{g}(\nu)m(\nu)}{4\pi},$$
$$\delta \tilde{g}(\nu) = \frac{15\tilde{h}(\nu)m(\nu)}{4\pi},$$
$$\delta \tilde{h}(\nu) = 0. \tag{46}$$

Accordingly, the bare parameters are given by

$$m_0^2 = m^2(\nu) + \frac{12\tilde{g}(\nu)m(\nu)}{4\pi},$$
$$\tilde{g}_0 = \tilde{g}(\nu) + \frac{15\tilde{h}(\nu)m(\nu)}{4\pi},$$
$$\tilde{h}_0 = \tilde{h}(\nu). \tag{47}$$

Since the bare parameters should be invariant under the change of the sliding renormalization scale, we obtain the following renormalization group equations

$$m^2(\mu) + \frac{12\tilde{g}(\mu)m(\mu)}{4\pi} = m^2(\nu) + \frac{12\tilde{g}(\nu)m(\nu)}{4\pi}, \tag{48}$$

$$\tilde{g}(\mu) + \frac{15\tilde{h}(\mu)m(\mu)}{4\pi} = \tilde{g}(\nu) + \frac{15\tilde{h}(\nu)m(\nu)}{4\pi},$$
$$\tilde{h}(\mu) = \tilde{h}(\nu). \tag{49}$$

We use these relations to get the parameters at scale $\nu = t \cdot \mu$ in terms of the parameters at a different scale $\mu$ as

$$m^2(\nu) = m^2(\mu) + \frac{12\tilde{g}(\mu)m(\mu)}{4\pi} - \frac{12\tilde{g}(\nu)m(\nu)}{4\pi},$$
$$\tilde{g}(\nu) = \tilde{g}(\mu) + \frac{15\tilde{h}(\mu)m(\mu)}{4\pi} - \frac{15\tilde{h}(\nu)m(\nu)}{4\pi}, \tag{50}$$
$$\tilde{h}(\mu) = \tilde{h}(\nu).$$

Plugging these to the OR equations yields

$$0=\left\{\begin{array}{l} m^2(\mu)+\frac{12\tilde{g}(\mu)m(\mu)}{4\pi}-\frac{\left(12\tilde{g}(\mu)+\frac{15\tilde{g}(\mu)m(\mu)}{4\pi}-\frac{15\tilde{g}(\nu)m(\nu)}{4\pi}\right)m(\nu)}{4\pi} \\ +4\left(\tilde{g}(\mu)+\frac{15\tilde{g}(\mu)m(\mu)}{4\pi}-\frac{15\tilde{g}(\nu)m(\nu)}{4\pi}\right)B^2+6\tilde{h}(\mu)B^4 \end{array}\right\}B, \tag{51}$$

$$0=\left\{\begin{array}{l} m^2(\mu)+\frac{12\tilde{g}(\mu)m(\mu)}{4\pi}-\frac{\left(12\tilde{g}(\mu)+\frac{15\tilde{g}(\mu)m(\mu)}{4\pi}-\frac{15\tilde{g}(\nu)m(\nu)}{4\pi}\right)m(\nu)}{4\pi}-t^2m^2(\mu) \\ \quad +12\left(\tilde{g}(\mu)+\frac{15\tilde{g}(\mu)m(\mu)}{4\pi}-\frac{15\tilde{g}(\nu)m(\nu)}{4\pi}\right)B^2+30\tilde{h}(\mu)B^4 \end{array}\right\}. \tag{52}$$

Then, by using the R-scheme relation $\frac{m(\mu)}{\mu}=\frac{m(\nu)}{\nu}$ as well as the dimensionless parameters $G=\frac{\tilde{g}}{4\pi m}$, $b=B\sqrt{\frac{4\pi}{m}}$ and $H=\frac{\tilde{h}}{(4\pi)^2}$, we get the following result

$$\begin{aligned} 0 &= \left(1+4G\left(3(1-t)+b^2\right)+6H\left(b^4+10b^2(1-t)-30t(1-t)\right)\right)b, \\ t^2 &= 1+12G\left((1-t)+b^2\right)+30H\left(b^4+6b^2(1-t)-6t(1-t)\right). \end{aligned} \tag{53}$$

For the vacuum energy density, Eq.(40) takes the form

$$\frac{8\pi E}{m^3}=b^2+2b^4G+2Hb^6+12b^2G(1-t)+30H(1-t)\left(b^4-6b^2t\right). \tag{54}$$

The solutions of Eq.(53) can be obtained numerically and the results of these calculations are presented in section IV.

## B. Two-loop calculations

Note that the one-loop divergent diagrams are cactus diagrams and thus can be renormalized using normal-ordering. In that case, one shall start with a Hamiltonian that is normal-ordered with respect to the vacuum of mass parameter $m$. Thus, we can use the relation [24]

$$N_m \exp(i\beta\psi)=\exp\left(-\frac{1}{2}\beta^2\Delta\right)N_{M=t\cdot m}\exp(i\beta\psi), \tag{55}$$

to obtain the Hamiltonian which is normal-ordered with respect to the new mass parameter $M=t\cdot m$. Here,

$$\Delta=\frac{1}{4\pi}(m-M). \tag{56}$$

In Eq.(55), expanding both sides and equating the coefficients of the same power in $\beta$ yields the result

$$\begin{aligned} N_m\psi &= N_M\psi, \\ N_m\psi^2 &= N_M\psi^2 + \Delta, \\ N_m\psi^3 &= N_M\psi^3 + 3\Delta N_M\psi, \\ N_m\psi^4 &= N_M\psi^4 + 6\Delta N_M\psi^2 + 3\Delta^2, \\ N_m\psi^5 &= N_M\psi^5 + 10\Delta N_M\psi^3 + 15\Delta^2\psi, \\ N_m\psi^6 &= N_M\psi^6 + 15\Delta N_M\psi^4 + 45\Delta^2\psi^2 + 15\Delta^3, \end{aligned} \tag{57}$$

and

$$N_m\left(\frac{1}{2}(\nabla\psi)^2 + \frac{1}{2}\Pi^2\right) = N_M\left(\frac{1}{2}(\nabla\psi)^2 + \frac{1}{2}\Pi^2\right) + \frac{1}{24\pi}\left(M^3 - m^3\right). \tag{58}$$

By applying the transformation $\phi \rightarrow \psi + B$ to the Hamiltonian in Eq.(33) with the use of Eqs.(57) and (58), we get

$$N_m H = \bar{H}_{0N} + \bar{H}_{IN} + \bar{H}_{ctN} + \bar{H}_{1N} + E, \tag{59}$$

where

$$\begin{aligned} \bar{H}_{0N} &= N_M\left(\frac{1}{2}\left(\Pi^2 + (\nabla\psi)^2\right) + \frac{1}{2}M^2\psi^2\right), \\ \bar{H}_{IN} &= \frac{g(\nu)}{4!}N_M\left(\psi^4 + 4B\left(\psi^3\right)\right) \\ &\quad + \frac{h(\nu)}{6!}N_M\left(\psi^6 + 6B\left(\psi^5 + 10\Delta\psi^3\right) + 15\psi^4(B^2 + \Delta) + 20B^3\psi^3\right), \end{aligned}$$

$$\begin{aligned} \bar{H}_{1N} &= \frac{1}{2}N_M\left(m^2(\nu) - M^2 + \frac{12g(\nu)}{4!}\left(B^2 + \Delta\right) + 30\frac{h(\nu)}{6!}\left(B^4 + 6\Delta B^2 + 3\Delta^2\right)\right)\psi^2 \\ &\quad + N_M\left(m^2(\nu) + \frac{4g(\nu)}{4!}\left(B^2 + 3\Delta\right) + \frac{6h(\nu)}{6!}\left(B^4 + 10\Delta B^2 + 15\Delta^2\right)\right)B\psi, \end{aligned} \tag{60}$$

and

$$\begin{aligned} E &= \frac{1}{2}\left(m^2(\nu) + \frac{12g\Delta}{4!}\right)B^2 + \left(\frac{g(\nu)}{4!} + \frac{15h\Delta}{6!}\right)B^4 + \frac{1}{2}\left(30\frac{h(\nu)}{6!}\left(3\Delta B^2 + \Delta^2\right)\Delta\right) \\ &\quad + \frac{h(\nu)B^6}{6!} + \frac{1}{24\pi}\left(M^3 - m^3\right) + \frac{3g(\nu)\Delta^2}{4!} + \frac{1}{2}m^2\Delta. \end{aligned} \tag{61}$$

According to the OR method, the Hamiltonian should be written in the correct form. In other words, $\bar{H}_{1N}$ has to be zero. Thus, putting the coefficients of $\psi$ and $\psi^2$ of $\bar{H}_{1N}$ equal to zero, respectively, yields

$$\left(m^2(\nu) + \frac{4g(\nu)B^2}{4!} + \frac{6h(\nu)B^4}{6!} + \frac{12g\Delta}{4!} + \frac{90h\Delta^2}{6!} + \frac{60h\Delta B^2}{6!}\right)B = 0, \tag{62}$$

and

$$\left(m^2(\nu) - M^2 + \frac{g(\nu)B^2}{2} + \frac{h(\nu)B^4}{4!} + \frac{h(\nu)}{8}\Delta^2 + \frac{g(\nu)}{2}\Delta + \frac{h(\nu)}{4}\Delta B^2\right) = 0. \tag{63}$$

In higher orders of loop expansions, we need only to consider diagrams which are not renormalized using the normal-ordering. Up to two-loops, we have the divergent diagrams shown in Fig.2. In the case of a scale transformation $\mu \to \nu = t\cdot\mu$, the coupling $g$ should be replaced by $\left(\frac{\mu}{\nu}\right)^{\epsilon} g_{new}$ and $g_{new} \to g$ as $\epsilon \to 0$ [20], where $\epsilon = 3 - d$.

To calculate $\delta m^2$, *i.e.* the two-loop counter term necessary for making the self-energy amplitude finite, we consider the first diagram in Fig.2. Using dimensional regularization, the counter term is obtained as

$$\delta m^2(\mu) = \frac{g^2}{3(8\pi)^2}\frac{1}{\epsilon}, \tag{64}$$

$$\delta m^2(\nu) = \frac{\left(\frac{\mu}{\nu}\right)^{2\epsilon} g_{new}^2}{3(8\pi)^2}\frac{1}{\epsilon} \approx \frac{g_{new}^2}{3(8\pi)^2}\frac{1}{\epsilon}(1 - 2\ln t). \tag{65}$$

The invariance of the bare mass under the scale transformation $\mu \to \nu = t\cdot\mu$ implies as $\epsilon \to 0$

$$m^2(\nu) = m^2(\mu) + \frac{g^2}{6(4\pi)^2}\ln t. \tag{66}$$

To calculate $\delta Z$, we look at the poles in the derivative of two-point function with respect to the four-mometum square $p^2$. For the Hamiltonian that we consider in (2 + 1) dimensions, there are no diagrams that contribute to $\delta Z$ up to two-loops, *i.e.* the derivative is finite.

For $\delta h$, we consider the second diagram in Fig.2. In fact, there are 9 different diagrams (due to the permutations of the external legs, *e.g.* s, t and u channels) that contribute to $\delta h$. Our calculation leads to

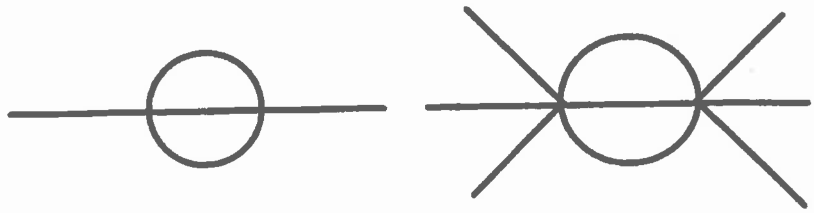

**Figure 2.** The two-loop divergent diagrames of the $\phi^6_{2+1}$ scalar field theory.

$$\delta h(\mu) = \frac{3h^2}{(8\pi)^2}\frac{1}{\varepsilon}, \tag{67}$$

$$\delta h(\nu) = \frac{3\left(\frac{\mu}{\nu}\right)^{2\varepsilon} h^2_{new}}{(8\pi)^2}\frac{1}{\varepsilon} \approx \frac{3h^2_{new}}{(8\pi)^2}\frac{1}{\varepsilon}\left(1 - 2\ln t\right). \tag{68}$$

Therefore, we obtain

$$h(\nu) = h(\mu)\left(1 + \frac{3h(\mu)}{2\,(4\pi)^2}\ln t\right). \tag{69}$$

As we get the parameters at scale $\nu$ in terms of those at scale $\mu$, we plug them into Eqs.(61), (62) and (63) to get

$$\left\{\begin{array}{l} m^2 + \frac{g^2}{6(4\pi)^2}\ln t - M^2 + \frac{4g}{4!}\left(B^2 + 3\Delta\right) \\ +6\frac{h\left(1+\frac{3h}{2(4\pi)^2}\ln t\right)}{6!}\left(B^4 + 10\Delta B^2 + 15\Delta^2\right) \end{array}\right\} B = 0, \tag{70}$$

$$\left\{\begin{array}{l} m^2 + \frac{g^2}{6(4\pi)^2}\ln t - M^2 + \frac{12g}{4!}\left(B^2 + \Delta\right) \\ +30\frac{h\left(1+\frac{3h}{2(4\pi)^2}\ln t\right)}{6!}\left(B^4 + 6\Delta B^2 + 3\Delta^2\right) \end{array}\right\} = 0, \tag{71}$$

and

$$E = \left\{\begin{array}{c} \frac{1}{2}\left(m^2(\mu) + \frac{g^2}{6(4\pi)^2}\ln t + \frac{12g\Delta}{4!} + \frac{90h\left(1+\frac{3h}{2(4\pi)^2}\ln t\right)\Delta^2}{6!}\right)B^2 \\ + \left(\frac{g(\nu)}{4!} + \frac{15h\Delta}{6!}\right)B^4 + \frac{h\left(1+\frac{3h}{2(4\pi)^2}\ln t\right)B^6}{6!} \\ +\frac{1}{24\pi}\left(M^3 - m^3(\mu)\right) + \frac{3g(\nu)\Delta^2}{4!} + \frac{1}{2}\left(m^2(\mu) + \frac{g^2}{6(4\pi)^2}\ln t\right)\Delta + \frac{1}{2}\left(\frac{30h}{6!}\Delta^3\right) \end{array}\right\} \tag{72}$$

Using the condition of fixed R-scheme, $\frac{m(\mu)}{\mu} = \frac{m(\nu)}{\nu}$, and the dimensionless parameters $G = \frac{g}{4\pi m}$, $b = B\sqrt{\frac{4\pi}{m}}$ and $H = \frac{h}{(4\pi)^2}$, we can rewrite Eqs.(70), (71) and (72), respectively, as

$$0 = \left\{ \begin{array}{c} 1 + \frac{G}{6}\left(3\left(1-t\right) + b^2\right) + \frac{H}{120}\left(b^4 + 10b^2\left(1-t\right) + 15\left(1-t\right)^2\right) + \frac{G^2}{6}\ln t \\ + \frac{H^2 \ln t}{80}\left(b^4 + 10b^2\left(1-t\right) + 15\left(1-t\right)^2\right) \end{array} \right\} b, \tag{73}$$

$$t^2 = \left\{ \begin{array}{c} 1 + \frac{G}{2}\left(\left(1-t\right) + b^2\right) + \frac{H}{24}\left(b^4 + 6b^2\left(1-t\right) + 3\left(1-t\right)^2\right) + \frac{G^2}{6}\ln t \\ + \frac{H^2 \ln t}{16}\left(b^4 + 6b^2\left(1-t\right) + 3\left(1-t\right)^2\right) \end{array} \right\}, \tag{74}$$

$$\frac{8\pi E}{m^3} = \left\{ \begin{array}{c} b^2 + \frac{G}{12}b^4 + \frac{H}{360}b^6\left(1 + \frac{3}{2}H\ln t\right) + \frac{1}{2}b^2 G\left(1-t\right) \\ + \frac{G^2}{6}\ln t\left(b^2 + \left(1-t\right)\right) + \frac{H^2(1-t)\ln t}{16}\left(b^4 + 3b^2\left(1-t\right) + \left(1-t\right)^2\right) \\ + \frac{G}{4}\left(1-t\right)^2 + \frac{1}{3}\left(t^3 - 3t + 2\right) + \frac{H(1-t)}{24}\left(b^4 + 3b^2\left(1-t\right) + \left(1-t\right)^2\right) \end{array} \right\}. \tag{75}$$

Note here that we use the original $\phi^4$ ($\phi^6$) coupling $\frac{g}{4!}\left(\frac{h}{6!}\right)$ rather than $\tilde{g}$ ($\tilde{h}$) to avoid large numbers coming at higher loop calculations. We use Eqs.(73), (74) and (75) to find the non-trivial duality-related solutions. The results are presented in the following section, Section IV.

# IV. Numerical calculations

## A. The symmetric (S) phase for one-loop calculations

For one-loop calculations, setting $b = 0$ in Eq.(53) yields the result

$$t^2 = 1 + 12G\left(1-t\right) + 30H\left(-6t\left(1-t\right)\right), \tag{76}$$

which is just the equation that constrains the mass to verify the invariance under the change of the sliding scale. This phase always (within one-loop calculations) has zero energy density.

Also, the above equation can be written in the form

$$M^2 = t^2 m^2(\mu) = m^2(\nu), \tag{77}$$

where $m^2(\nu)$ satisfies Eq.(48). Solving Eq.(76), we get $t = 1$ for the original phase and

$$t = \frac{2\alpha + 1}{-1 + 2\beta} \tag{78}$$

for non-trivial but symmetric phase, where $\alpha = 6G$ and $\beta = 90H$. For the consistency of our calculation, we restrict ourselves to the small $h$ region and thus $\beta = \frac{90h}{(4\pi)^2}$ is taken to be less than 1. This choice of $\beta < 1$ is to make the loop expansion consistent in the level of the truncation that we make. Thus, all the calculations in this work are carried out for small values of $\beta$. However, the S-phase can exist only for $\alpha$ given by

$$\left\{ \begin{array}{cc} \emptyset & \text{if } \beta = \frac{1}{2} \\ \left(-\infty, -\frac{1}{2}\right) & \text{if } \beta < \frac{1}{2} \\ \left(-\frac{1}{2}, \infty\right) & \text{if } \frac{1}{2} < \beta \end{array} \right\}. \tag{79}$$

This phase has a zero energy density (with one-loop). If there exists another phase (among the broken-symmetry phases) with smaller energy density, then of course the phase with the lowest energy density would be the preferred one by nature.

## B. The broken-symmetry (BS) phase for the one-loop calculations

In one-loop level and $b \neq 0$, we need to solve the following equations

$$\begin{aligned} t^2 &= 1 + 12G\left((1-t) + b^2\right) + 30H\left(b^4 + 6b^2(1-t) - 6t(1-t)\right), \\ 0 &= 1 + 4G\left(3(1-t) + b^2\right) + 6H\left(b^4 + 10b^2(1-t) - 30t(1-t)\right), \end{aligned} \tag{80}$$

for $b$ and $t$. There exist three different non-trivial solutions. All of them minimize locally the energy density:

$$\frac{8\pi E}{m^3} = b^2 + 2b^4G + 2Hb^6 + 12b^2G(1-t) + 30H(1-t)\left(b^4 - 6b^2t\right). \tag{81}$$

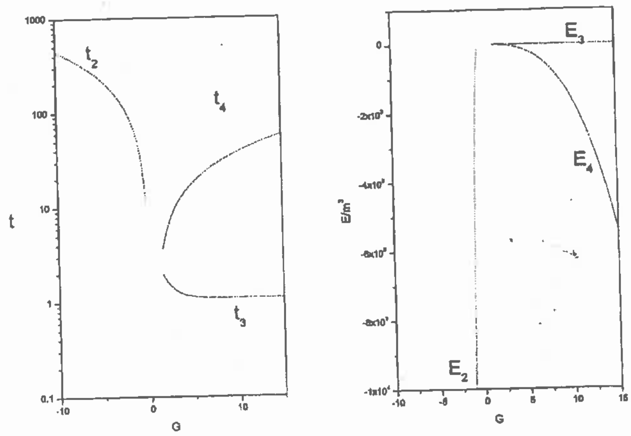


**Figure 3.** The non-trivial BS phases and the corresponding energy densities for $(\phi^4 + \phi^6)_{2+1}$ theory with $\beta = 0.01$, for one-loop calculations.

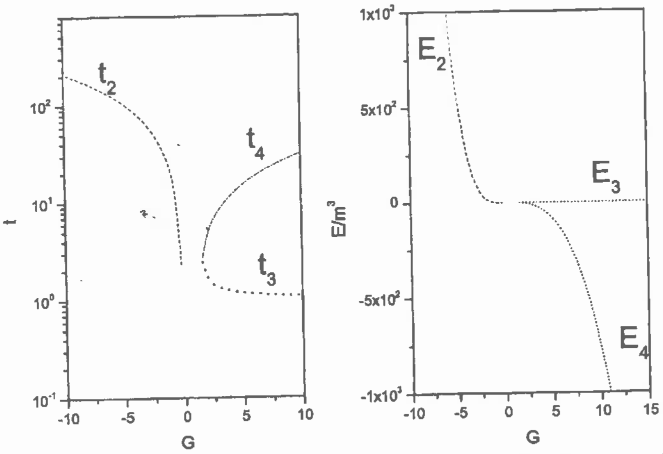


**Figure 4.** The non-trivial BS phases and the corresponding energy densities for $(\phi^4 + \phi^6)_{2+1}$ theory with $\beta = 0.08$, for the one-loop calculations. For negative $G$, the symmetry is restored.

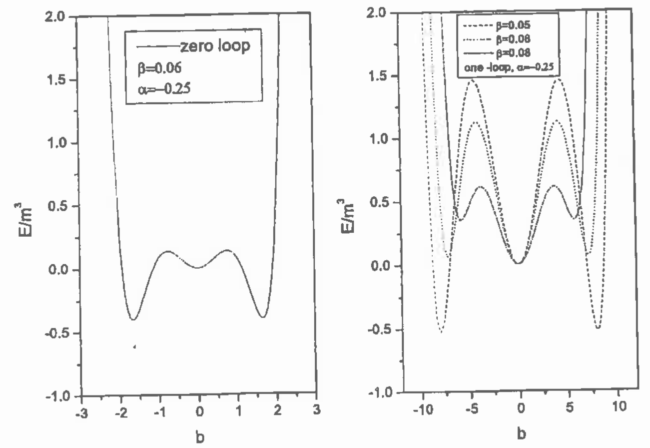


**Figure 5.** The vacuum energy density as a function of the classical field b, for both the classical (zero-loop) and one-loop cases.

Among the different solutions (phases), nature will choose the one which provides the smallest vacuum energy and the effective couplings. In Fig.3, the non-trivial solutions for the $t$ parameter and the corresponding energies are shown. In this figure, the phase that has the lowest energy is the $t_2$ phase. As we increase the $\beta$ parameter as shown in Fig.4, the phase $t_4$ is developed for the positive values of the $G$ parameter and has the smallest value of the energy density. This indicates that the symmetry which was broken at the classical level is restored for $G < 0$ as the $\beta$ parameter gets greater than 0.08. Also the symmetry restoration can be realized from Fig.5 for the Effective Potential. In fact, the solutions which minimize the energy density are the duality-related solutions, $t_2$ and $t_4$. We checked the effective coupling for those solutions and found that it indeed becomes perturbative for all available values of $\beta$.

## C. Two-loop calculations

At two-loop level in $(\phi^4)_{2+1}$ ($h$ = 0) theory, we reproduce the previous result of Ref.[7] as shown in Fig.6. However, our result indicates that the non-trivial symmetric phase is the preferred phase in contradiction with the case of Ising model. We believe that the situation here is due to the nonperturbative character of the theory at the critical coupling region. This situation may be improved by using a Borel-Sum as we recently apply for $(\phi^4)_{1+1}$ case[23]. Although our result agrees well with the $(\phi^4)_{2+1}$ prediction in Ref.[7], we note that the critical coupling in Ref.[7] is defined somewhat differently from ours. While we define the critical coupling by comparing the energy densities of the vacuum solutions, in Ref.[7] the critical coupling was defined by the coupling constant which yields the non-trivial vacuum solutions without considering the energy densities. Our definition is consistent with the one presented in Ref.[8], where it was shown that the prefered phase based on the energy density consideration is the S phase and the critical coupling $G_c$ was predicted at the point t=1 in the S phase. We calculated $G_c$ for the $(\phi^4)_{2+1}$ case using this definition and found it to be 5.28 which agrees with the result in Ref.[8] as it should be. However, neither our result nor the results in Refs.[7, 8] for the $(\phi^4)_{2+1}$ theory (for $G$ positive) is in agreement with universality arguments from the reasoning that the $\phi^4$ theory belongs to the same class of universality with the Ising model for positive $G$. At the critical point, the mass parameter approaches zero value which makes the effective coupling $G/M$ blow up. Therefore, perturbative calculations fail and a Borel-Sum may be necessary for an improvement[23].

For the $(\phi^6)_{2+1}$ theory, if $G$ is negative and $H = 0.08$ ($H \equiv 8\beta$ used in one-loop), there exist one non-trivial symmetric solution labeled as $S_1$ and a broken symmetry solution labeled as $BS_1$ in Fig.7. Also, for $G$ positive, the calculation predicts two phases with different symmetry properties ($S_2$ and $BS_2$). The corresponding energy densities (the

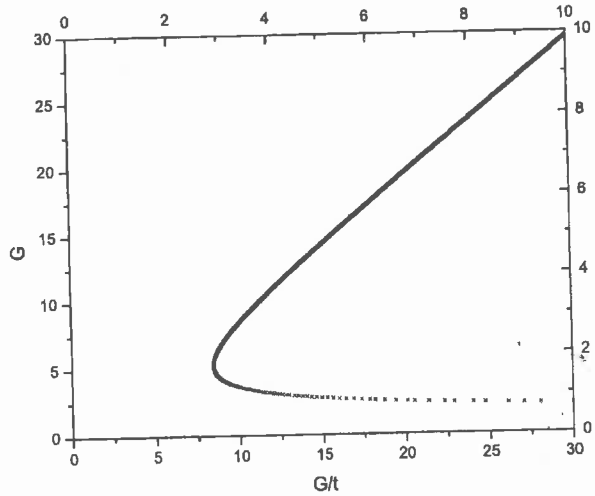


**Figure 6.** The original coupling $G$ plotted versus the effective coupling $G/t$ for the $\phi^4_{2+1}$ including the two-loop calculations.

right panel of Fig.7) show that the preferred phase for negative (positive) $G$ values is $BS_1$ ($S_2$). Increasing the value of $H$ to $H = 0.37$, the symmetry is restored for negative $G$ and no symmetry breaking exists for positive $G$. This can be realized from Fig.8, where we plot the preferred phases only. For negative values of $G$, the result presented in Fig.9 shows the first order phase transition as expected[25]. This result distinguishes from the GEP result in Ref.[12] where a second-order phase transition was found at small values of the coupling constants $G$ and $H$. In fact, the success of the OR method to predict the correct phase transition for negative $G$ values is due to the duality. For instance, in the first order phase transition, the mass parameter never approaches small values and thus the effective coupling stays small. To illuminate duality in our calculations, we plot the classical effective potentials in Fig.10. The equivalence of the effective potentials for two phases with different symmetry properties is easily realized. This is due to the fact that for very large value of the original coupling $G$, the effective coupling for the non-trivial phases is small and thus quantum corrections are negligible.

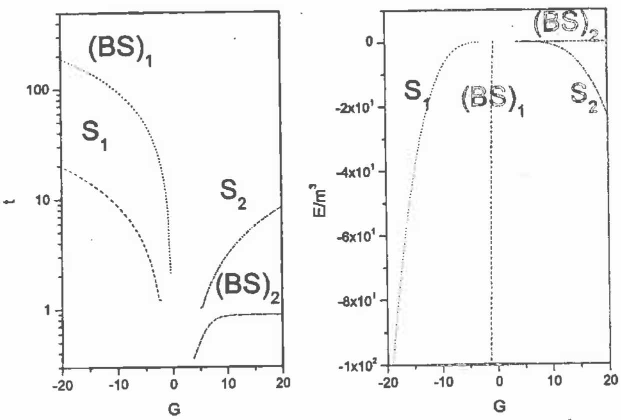


**Figure 7.** The $t$ parameter and the corresponding energy density versus the coupling $G$ for $H$ = 0.08 (two-loop calculation).

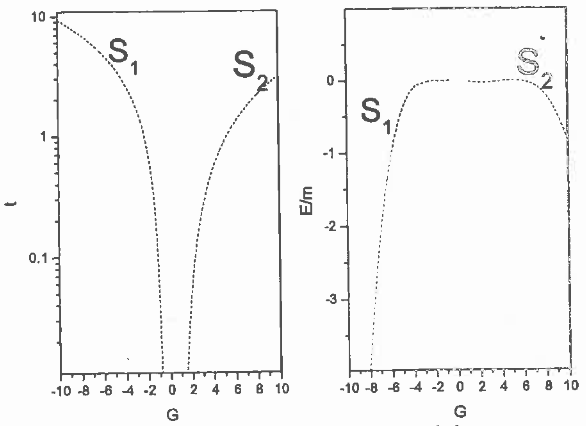


**Figure 8.** The non-trivial and symmetric phase parameter $t$ and the corresponding energy density versus the coupling $G$ for $H$=0.37 (two-loop calculation).

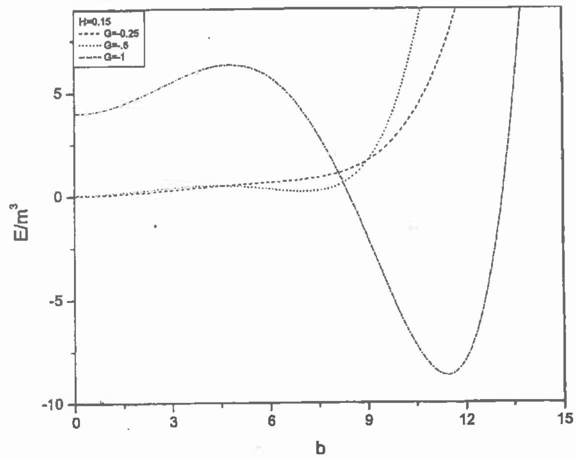


**Figure 9.** The effective potential for the $\phi^6_{2+1}$ (two-loop calculation)

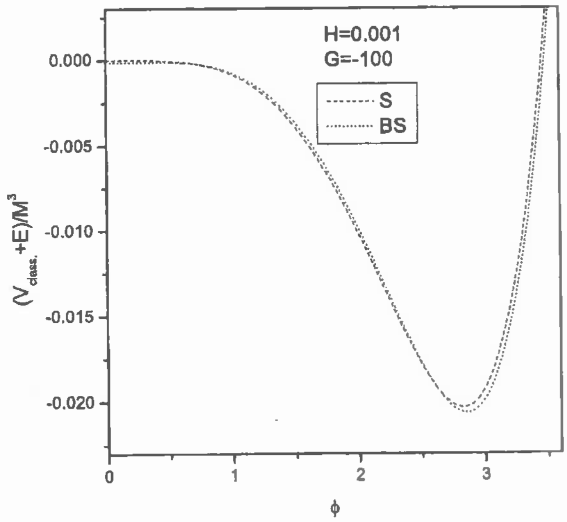


**Figure 10.** The classic effective potential for both $S$ and $BS$ phases for large $G$ value within the two-loop calculations.

## V. Summary and conclusion

We applied the OR method to study the phase structure of $\phi^6$ model in (2+1) dimensions. From our application, we realize that the original theory of strong interactions can be solved perturbatively using the equivalent quasiparticle theory. The key to achieve this result is the duality relations between the original theory and the quasiparticle theory.

The OR method uses a very simple assumption; *i.e.*, the interaction Hamiltonian should only include terms that have powers greater than two in the ld operator after a canonical transformation. We show that with the specifc canonical ınsformation of field shift this method is a manifestation of the Effective Action rmulation. Accordingly, unlike the HA and the GEP, the OR method does not suffer m renormalization problem in higher dimensions[26]. Moreover, the OR method may regarded as more general than the Effective Action method since a variety of ɔosing canonical transformations can be available in the OR method. Also, the OR :thod allows the prediction of the symmetry restoration while GEP [3] calculations ulted in an unstable Effective Potential which prevent GEP from studying the nmetry restoration. Our calculations show that the phase transition (for negative $G$) is the first order, which is expected[25] for the kind of potential we have. At the same ıe, this result distinguishes from the GEP in Ref.[12], where a second-order phase nsition occurring at small values of coupling constants was found. The two-loop lculation for the $(\phi^6)_{2+1}$ theory shows that the symmetry is restored for negative $G$ as : increase $H$ up to the value $H_c \approx 0.37$, or $\beta = 0.04625$ (compared to $\beta = 0.08$ in one-ɔp). Improvements of our calculations near the critical points are in progress using a ɔrel-Sum technique [23].

Although the method turns out to be quite successful for the low dimensional super-ıormalizable theory, it is not yet clear how it can be applied to the renormalizable ɛories in 3+1 dimensions. We hope that the use of a Borel-Sum technique would be eful in this direction.

## .cknowledgments

We thank Dr. Andrey Vinnikov and Prof. Dong-Pil Min for many helpful scussions. This work was supported by the DOE under contract DE-FG02-96ER40947.

## :eferences


R. Rajaraman and M. Raj Lakshmi, Phys. Rev. D**23**, 2399 (1980).
J. Casahorran, J. C. Ciria, J. G. Esteve and A. Tarancon, Phys. Lett. B**303**, 99 (1993).
Wen-FA Lu, Guang-Jiong and Zhi-Guo Wang, J. Phys. G: Nucl. Part. Phys. **24**, 673 (1998).
J. M. Cornwall, R. Jackiw and E. Tomboulis, Phys. Rev. D**10**, 2428 (1974).
G. V. Efimov, Int. J. Mod. Phys. A**4**, 4977 (1989).
S. J. Chang, Phys. Rev. D**13**, 2778 (1976) [Erratum-ibid. D**16**, 1979 (1976)].
S. F. Magruder, Phys. Rev. D**14**, 1602 (1976).
M. Dineykhan, G. V. Efimov, G. Ganbold and S. N. Nedelko, Lect. Notes Phys. M**26**, 1 (1995).
P.M. Stevenson, Phys. Rev. D**30**, 1712 (1984).
). P.M. Stevenson, Phys. Rev. D**35**, 2407 (1987).
l. P.M. Stevenson, Phys. Rev. D**32**, 1389 (1985).
2. P. M. Stevenson and I. Roditi, Phys. Rev. D**33**, 2305 (1986).
3. Paolo CEA , Phys. Lett. B**236**, 191 (1990).
4. Paolo CEA , Phys. Lett. B**335**, 432 (1994).
5. H. Mishra and A. R. Panda, J. Phys. G : Nucl. Part. Phys. **18**, 1301 (1992).
6. S. J. Chang, Phys. Rev. D**12**, 1071 (1975).
7. S. J. Chang and J. A. Wright, Phys. Rev. D**12**, 1595 (1975); S. J. Chang and T.-M. Yan, Phys. Rev. D**12**, 3225 (1975).
8. C.-R. Ji, J. I. Kim, D. P. Min and A.V. Vinnikov, arXiv:hep-ph/0204114, v2.
9. Book by Michael E.Peskin and Daniel V.Schroeder, "AN INTRODUCTION TO THE QUANTUM FIELD THEORY", Addison-Wesley Publishing Co. (1995).

20. Book by John C. Collins "RENORMALIZATION", Cambridge University Press (1984).
21. In both normal-ordering and loop expansion calculations, the OR equations are the same while the energy density differs by a field independent term which does not affect the critical behavior.
22. C.-R. Ji and Y. Mishchenko, "Oscillator Representation Method for the $(\phi^4)_{1+1}$ Theory in the Presence of External Magnetic Field", in preparation.
23. A. Shalaby, Y. Mishchenko and C.-R. Ji, "Borel-Summation and Oscillator Representation Method in the $(\phi^4)_{1+1}$ Theory", in preparation.
24. S. Coleman, Phys. Rev. D**11**, 2088 (1975).
25. Salman Habib and Avadh Sakena, Proceeding of NEED's94, Los Alamos, September(1994).
26. U. Ritschel, Z. Phys. C 54, 297 (1992), J. Wudka, Phys. Rev. D **37**, 1464 (1988).