Kyiv National University of Taras Shevchenko

The Faculty of Physics

Mishchenko, Yuriy

Nonperturbative Mass Renormalization in 2+1 Scalar Yukawa Model

(Thesis for the Completion of M.Sc. Degree)

Academic Advisor:
Prof. Dr. Simenog, Ivan
(Head of the PPTF department at the Institute for Theoretical Physics of N. Bogolyubov at the National Academy of Sciences of Ukraine)

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ABSTRACT

The thesis considers the problem of nonperturbative description of single- and two-particle states in the scalar Yukawa model using variational approach in Hamilton formulation of quantum field theory. In the limit of weak and strong coupling, light and heavy nucleons, the thesis produces analytical relationships for renormalization of the mass of a single nucleon. The thesis also produces the effective equation for the coupling energy of two nucleons in the limits of weak and strong coupling. The thesis also develops a framework for numerically calculating the effect of mass renormalization in 1+1 for all values of the coupling constant, and performs numerical calculations of bare mass for different values of the coupling and the physical mass.

Contents

1. Introduction

Study of many-particle states in quantum field theory is an important problem for many areas of physics. Such problems appear in quantum electrodynamics, atomic physics, nuclear physics, and elementary particle physics. Various methods exists in modern theoretical physics to solve such problems.

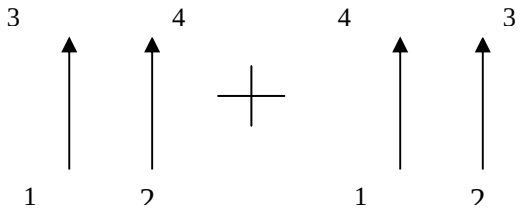
The most consistent and developed of such methods is the approach of the Bethe-Solpeter equations. In the framework of this approach, considering a two-particle state consists in using the respective Green function (propagator) [2,16,37]:

$$G^{(2)}(x_1 - x_3, x_2 - x_4) = \langle 0 | T[S\psi(x_3)\psi(x_4)\overline{\psi}(x_1)\overline{\psi}(x_2)|0\rangle_c \qquad (1)$$

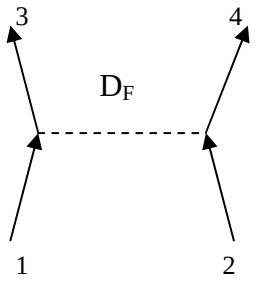
The main idea subsequently consists in construction of the equation for $G^{(2)}$ via the single-particle Green functions $G^{(1)}(x-y) \equiv G(x,y)$. This program can be realized within the approach of the theory of perturbations – the process of particle propagation is represented as a sum of all possible interaction diagrams, and these diagrams are summed together as:

$$\begin{aligned} G^{(2)}(12;34) &= G(13)G(24) + G(14)G(23) - e^2/2\times \\ &\times \int dx_5 dx_6 G(15)G(53)D_F(56)G(26)G(64) + \ldots \end{aligned} \qquad (2)$$

Here, the first term describes “free” propagation of the particles:



while the second term describes their interaction in the first order of the theory of perturbation:



Using the complete perturbative series the final equation can be constructed:

$$G^{(2)}(12;34) - G(13)G(34) = i\int dx_5 dx_6 dx_7 dx_8 G(35)G(46)R(x_1, x_2; x_3, x_4)G^{(2)}(78;12),$$

where $R(x_1, x_2; x_3, x_4)$ – is a specific sum of the connected diagrams from (2).

For bound states two-particle propagator is represented using the product of the amplitudes $\chi_{p\alpha}(12)=\langle 0|T(\psi(1)\psi(2))|p\alpha\rangle$, where $|p\alpha\rangle$ is the bound state, and for these a relativistic wave equation can be obtained in the form:

$$\chi_{pb}(12)=-i\int dx_5 dx_6 dx_7 dx_8 G(15)G(26)R(56;78)\chi_{pb}(78). \qquad (3)$$

which is the Bethe-Salpeter equation.

Despite consistency and covariant nature of equation (3), as well as the possibility of its solution via the theory of perturbations, working with this equation is typically very difficult. As we can see, this is a space-time integral equation, which makes solving this equation very difficult. Furthermore, this equation has difficulties with multiple products of δ-functions as well as ultraviolet divergences in integrals, which are also present in other field theories. Finally, the amplitude χ, for which the above equation was constructed, does not have a transparent physical interpretation such as, for example, the wave function in the Schrodinger equation. These difficulties make application of Bethe-Salpeter equation by far a non-trivial exercise.

In fact, due to these difficulties with relativistic field-theoretic equations of the type of Bethe-Salpeter, a widely used practice in theoretical particle physics is to use various potential models and effective two-particle states similar to Schrodinger equation [15,19,35]. In this approach, the description equations are constructed as integro-differential equations including kinetic and potential terms. The kinetic term normally is generalized to a relativistic form (for example, $p^2/2m\to\sqrt{p^2+m^2}$), in order to described relativistic effects, and the potential is calculated from the quantum field theory or is modeled via some other means. Naturally, such model or effective potentials are required to have the properties present in the respective field theory (for example, the spin structure) or the phenomenology (for example, confinement).

Some of the best-known examples of this approach is construction of the effective potential within the framework of quantum electrodynamics (that is, Coulomb force followed by relativistic and spin corrections) or construction of the effective quark interaction potentials (short-range $\approx 1/r$ interaction is obtained from perturbation theory and long-range $\approx r$ interaction is introduced from phenomenology in order to represent confinement) [35]. In nuclear physics, where hadron-hadron interaction potential cannot be obtained consistently from the first principles, a widely used practice is the use of phenomenological (effective) potentials [19]. This approach, of course, is by far simpler than the Bethe-Salpeter equa-

tions. On the other hand, this approach has the problem of connecting the effective potentials with fundamental field theory in the relevant physical settings.

One of the solutions for this problem is the use of the perturbation theory, where the potential is constructed from several first terms in perturbation series. However, such approach cannot be used in nuclear physics where the coupling constant is not small. In these settings, one has to sum up in a certain way all orders of the perturbation series and use a nonperturbative scheme such as path integrals or dispersion relations.

One of such nonperturbative schemes can be constructed starting with the variational principle in Hamilton formulation of quantum field theory. This method allows one to consider processes in the format close to the traditional Schrodinger equation, and can be easily and naturally generalized to the case of three and more particles. Also, variational method can be used in strongly-bound problems. As a drawback of this approach – the quality of produced results directly depends on the good choice of the class of variational functions.

Some of the first works using variational approach in quantum field theory date back to 60th, where the variational approach in these settings was first considered by Schiff [20] and subsequently developd by Coleman, Barnes, Stevenson in 70-80th [1,3,23,24]. Variational method has been actively employed in various problems relating to atomic, nuclear, and quark systems [5,9-13,21,22,26,27].

Hamilton formulation of QFT is based on the relation $:H:\left|\Psi\right\rangle = E\left|\Psi\right\rangle$, that can be also re-written in covariant form as

$$P^{\mu}P_{\mu}\left|\Psi\right\rangle = M^{2}\left|\Psi\right\rangle \tag{4}$$

(here P^{μ} is the quantum energy-momentum operator). In the frame of the center of mass where $\vec{P}\left|\Psi\right\rangle = 0$, (4) reduces to $:H^{2}:\left|\Psi\right\rangle = M^{2}\left|\Psi\right\rangle$, which is equivalent to (4) except for the exclusion of the negative energies sector. The same equation can be also rewritten as

$$\left\langle \delta\Psi\right|:H-E:\left|\Psi\right\rangle = 0 \tag{5}$$

(or $\delta\left\langle \Psi\right|:H-E:\left|\Psi\right\rangle = 0$), which is equivalent to (4) assuming free variability of $\delta\Psi$.

In practice, of course, this latter condition cannot be realized, and the calculation of variations is performed on a specific class of states that have the characteristics relevant for the problem at hand. The freedom in the choice of the class of variation functions and their parameters is at the same time the advantage and the drawback of the variational approach. On one hand, this allows to formulate the problem at such a level of complexity where it can be analyzed and solved. On the other hand, if chosen class of variational functions is not

sufficiently close to reality or is not sufficiently rich, variational analysis is unlikely to produce a useful result. The problem of the choosing realistic yet analytically tractable state-anzatses in quantum field theories can be especially difficult given the complexity of the field theory.

We shall also note that variational approach realized in Fock space is similar to the method of Tamm, even though the latter is not formulated variationally [4,25]. The chains of the equations of the type of Tamm can be obtained in the following way. The state of the system is described in the Fock state using a superposition of many-body states as follows,

$$\begin{aligned} &\left|\Psi_N\right\rangle = \sum_{n=0}^{\infty} \frac{g^n}{n!} \int dp_1 \ldots dp_N \int dq_1 \ldots dq_n c^{(N;n)}_{\{p\},\{q\}} b^+_{p_1} \ldots b^+_{p_N} a^+_{q_1} \ldots a^+_{q_n}; \\ &\left|\Psi\right\rangle = \sum_{N=1}^{\infty} \left|\Psi_N\right\rangle \end{aligned} \quad . \qquad (6)$$

After that, this "anzats" can be used in Hamilton formalism $: H : \left|\Psi\right\rangle = E\left|\Psi\right\rangle$, where :H: – normal-ordered Hamiltonian. This substitution results in a chain of coupled integral equations for Fock amplitudes $c^{(N;n)}_{\{p\},\{q\}}$. The equations are coupled both via the states with different number of particles *b* as well as via the states with different number of particles *a*. At the same time, if the Hamiltonian conserves the number of particles of a certain sort (for example *a*), such as in the model of heavy nucleon, then the equations do not couple via that parameter and can be split into separate chains and separate equations for the Fock states with different by definite number of particles *a*.

One of the classes of the problems where variational method found successful applications is the problem of the structure and the stability of the vacuum state with respect to sinle-particle perturbations in scalar field theories, which other than their simplicity are also directly connected with the Higgs' sector of the Standard Model [1,3,23,24]. The Lagrangian in this case is chosen as a sum of scalar kinetic and potential terms in the form $V(\varphi) = \frac{1}{2} {m_0}^2 \varphi^2 + \lambda_0 \varphi^4 + \eta_0 \varphi^6$ for one, two, and three-dimensional space plus time. The class of variational functions is chosen in the form of the superposition of free-states. Fpr example, two-body state is described in this form,

$$\left|\Psi_2\right\rangle = \int d\vec{p} \cdot f(\vec{p}) a^+_p a^+_{-p} \left|0\right\rangle$$

Substitution of this form to (5) produces an integral equation for the amplitude *f*, which can be solved numerically or analytically perturbatively [7,8].

Another class of interesting problems is the model of Yukawa based on the Lagrangian:

$$L = \partial_\mu \varphi^* \partial^\mu \varphi - {M_0}^2 \varphi^* \varphi - g\phi^* \varphi\chi - \lambda(\varphi^* \varphi)^2 + \frac{1}{2}\left(\partial_\mu \chi \partial^\mu \chi - {m_0}^2 \chi^2\right). \qquad (7)$$

In this case different processes are considered using approximate variational anzatses. For example, particle-antiparticle system that decays by emitting two χ-particles can be inspected using the class of variational functions of the form $|\Psi_2\rangle = F|\bar{\varphi}\varphi\rangle + G|\varphi\bar{\varphi}\chi\rangle + A|\chi\chi\rangle$, which produces three integral equations for form-factors *F, G, A* [11]. Solution of these three equations, either numerical or approximate analytical, is complex. For this reason researchers typically exclude from these two out of three forma-factors using perturbation theory, obtaining a single equation for *F* that describes the relative motion of the particles φ in the system.

Similar approach is used in [13,14] for description of the bound state of positronium e^+e^- in QED. Variational anzats in this case was chosen as:

$$|\Psi\rangle = F\left|e^+e^-\right\rangle + G_1\left|e^+e^-\gamma\right\rangle + G_2\left|e^+e^-e^+e^-\gamma\right\rangle + G_3|\gamma\rangle,$$

where all terms *G* were excluded approximately in order to formulate the final equation for *F* that describes the motion of leptons up to order α^3. In the same way, the decay of positronium into a pair of photons can be stated, as well as a large number of other related problems from electrodynamics to chromodynamics of quark-antiquark systems (that is meson spectroscopy) [27].

On other interesting application of variational approach in QFT is formulation of nonperturbative equations using anzats of the sort [9,10,12,22]:

$$|\Psi_2\rangle = F_0|\varphi\bar{\varphi}\rangle + F_1|\varphi\bar{\varphi}\chi\rangle + F_2|\varphi\bar{\varphi}\chi\chi\rangle + \ldots \qquad (8)$$

Such approach, differently from earlier examples where the accuracy of the results in *g* is set by the number of terms in variational anzats, allows one to perform the calculations consistently regardless of the smallness of the coupling constant *g*. In [9] this idea had been realized for a system of spinor fields that represented real nucleons interacting via π-meson exchanges.

In [21] similar problem is considered for Yukawa model of scalar fields using Tamm equations. Although it is not possible to solve the entire chain of Tamm equations, certain assumptions were made regarding the amplitudes $c^{(N;n)}_{\{p\},\{q\}}$ or the chain is truncated at certain depth. In particular, truncation of the chain allows to obtain the equations for mass renor-

malization as well as obtain two-particle relativistic wave equation. In the approximation of heavy nucleon, using certain substitution, the equations can be split exactly. However, it appears that such simplification is equivalent to Hamiltonian diagonalization with a simple unitary transformation, and in this case does not produce interesting dynamics. Furthermore, although the above work begins with a consistent variational approach, later on various approximations are applied destroying the rigorousness of the calculations.

Similar program executed in somewhat more principled and rigorous manner is realized in [9], where mass renormalization and two-body equations are similarly constructed. As far as it is necessary for defining the problem in this thesis, the contents of that work are briefly described in next section. In this thesis, we will carry out numerical calculations using the simplest form-factor for the case of one dimension. Subsequent development of this program is performed in [22], where numerical approach for solving such problems with controlled precision is developed.

This work develops and applies the above framework to two-dimensional scalar Yukawa model, which is a model of nucleon-nucleon interactions for π-meson exchanges. This model is interesting not only from the point of view of nonperturbative description of nuclear interactions, derivation of the wave equations and effective potentials for description of systems with many nucleons, but also from methodological standpoint, since it allows to consider the nature of the physical interactions with vacuum, renormalization, and may further our understanding of divergences in modern interacting quantum field theories.

This thesis also performs derivation of the properties of single- and two-particle states in this model, for which a numerical program is developed for calculation of the effect of renormalization for intermediate values of the coupling. This numerical program together with analytical results for strong and weak interaction provides complete solution of the renormalization problem in Yukawa model for any coupling strength and high precision.

2. The model of heavy nucleon

A model similar to that considered in the thesis is the model of heavy nucleon [32,37]. This model describes the Yukawa-type interaction of light bosons (mesons π) with mass *m* and dispersion law $\omega_k = \sqrt{k^2 + m^2}$ and heavy fermions (nucleons N) with mass *M*. The fermions are considered to be heavy so that their energy's dependence on momentum as well as spin can be neglected. The Hamiltonian of such a system is described in terms of two fields with interaction $N \leftrightarrow N + \pi$, where the efficiency of such conversion is moment-dependant and is modeled by a nonlocal form-factor. Using the operators of creation and annihilation of the respective particles, the Hamiltonian can be written in the following form:

$$\begin{aligned} H &= H_0 + H_I \\ H_0 &= M\int d\vec{p} a_p^+ a_p + \int d\vec{q}\cdot\omega_q b_q^+ b_q, \\ H_I &= \int d\vec{p}\int d\vec{k} g(k)\omega_k a_{p+k}^+ a_p \left(b_k + b_{-k}^+\right) \end{aligned} \tag{9}$$

where the function $g(k)$ is the interaction form-factor. It is important that non-locality of the interaction here is postulated, and so can be used to compensate the ultraviolet divergences that appear in the limit $\left(g(k) = \frac{g}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega_k{}^3}} \right)$. This Hamiltonian commutates with the operator of total momentum and the number of nucleons, therefore its eigenstates can be described by these quantum numbers.

The most important property of this model is that it admits analytical solution. In particular, the single-nucleon problem in this model (renormalization of nucleon mass) has the form:

$$\begin{aligned} N_a\Phi_1 &= \int d\vec{k} a_k^+ a_k \Phi_1 = 1\cdot\Phi_1 \\ P_\mu\Phi_1 &= p_\mu\Phi_1 \end{aligned}, \tag{10}$$

or in other words $\vec{P}\Phi_1 = \vec{p}\cdot\Phi_1; H\Phi_1 = M\cdot\Phi_1$. Departing from (10), the state with the total momentum *p* can be written as:

$$\Phi_{1,\vec{p}} = \int d\vec{q}\Phi_0(\vec{q}) a_{p-q}^+ \tag{11}$$

where $\Phi_0(q) = F_b(q)|0\rangle$ is the state without nucleons but with an arbitrary number of mesons, $F_b(q)$ is a certain functional depending on the creation operators b_q^+ with total

momentum q. Calculating the commutator of (10) with a_0, we obtain the following integral equation for the amplitude:

$$\left(M + \int d\vec{k}\,\omega_k b_k^+ b_k\right)\cdot \Phi_0(p) + \int d\vec{k}\cdot g(k)\omega_k\left(b_k + b_k^+\right)\cdot \Phi_0(p+k) = E\cdot\Phi(p) \qquad (12)$$

For subsequent analysis the following Eq. (13) for operator $A = A^+ + A^- = \int d\vec{k}\left(\lambda_+(k)a_k^+ + \lambda_-(k)a_k\right)$ [32] can be used:

$$\left[n_k, e^{\pm A}\right] = \left[\pm K(k) + \lambda_+(k)\lambda_-(k)\right] e^{\pm A}, \qquad (13)$$

where

$$[n_k, A] = \left[a_k^+ a_k, A\right] = \lambda_+(k)a_k^+ - \lambda_-(k)a_k = K(k) \qquad (14)$$

Noticing that operator $K(k)$ in (14) in fact has the same form with the interaction-term in our equation, we can write the solution in the form $\Phi_0(p) = F_p|0\rangle = \int dx e^{ipx} e^{A(x)}|0\rangle$. The commutator with the total meson-energy operator $W = \int d\vec{k}\cdot\omega_k b_k^+ b_k$ gives

$$\left[W, F_p\right] = \int dk dx\,\omega_k\left(\lambda_+(k)b_k^+ - \lambda_-(k)b_k\right)e^{ipx}e^{A(x)} + \int dk\,\omega_k\lambda_+(k)\lambda_-(k)e^{A(x)} \qquad (15)$$

and noticing $\lambda_\pm(k) = \pm e^{\mp ikx} g(k)$ we obtain:

$$\begin{aligned} &\left[F_p, W\right] = \int dk\cdot\omega_k g(k)\left[b_k + b_{-k}^+\right]F_{p+q} + \Delta M\cdot F_p, \\ &\Delta M = \int dk\cdot\omega_k g^2(k) \end{aligned} \qquad (16)$$

From here, taking into account that $W|0\rangle = 0$, we obtain for Φ_0 the following equation:

$$W\Phi_0(p) + \int dk\cdot g(k)\omega_k\left[b_k + b_k^+\right]\Phi_0(p+k) + \Delta M\cdot\Phi_0(p) = 0, \qquad (17)$$

That is, this is the solution of the model for single nucleon and the quantity ΔM is the renormalization of the mass of the nucleon.

This model allows to analytically consider the problem of the renormalization of mass and the real form of the single-nucleon state in field theory. An important property of thus obtained result is that the terms with several meson-creation operators contain the amplitude which is factorized with respect to their moments, $c^{(1,n)}_{p;q_1,q_2\ldots,q_n} = g(q_1)\ldots g(q_n)$. Infinite sums of this form have the name of coherent states, which in another way can be defined as the eigenstates of the meson-annihilation operator b_0. As in this calculation, in this thesis we also use coherent states to construct nonperturbative single- and two-particle states in Yukawa model under variational approach.

The Hamiltonian of the above model can be also linearized using the substitution $c(k) = b(k) + v(k)$, where $v(k)$ is a regular c-function. This is just another reflection of exist-

ence of exact analytical solution of this problem. Finally, in the limit of local interaction, the renormalization of mass produces renormalization divergences in 3D space.

There exist also some other models in the literature that admit exact solutions, such as the model of Li (several particle types), model of Ventzel (scalar field with non-point particle source), model of Ruigrok ve Van-Hoff (several conversion processes) and other [37]. At the same time, in terms of its processes the heavy nucleon model is close to the problem that will be considered in this work, in this respect useful for deeper understanding of the problem, analytical, and numerical results considered in this thesis.

3. Problem formulation

The main advantage of simple quantum field theoretic models is in that that they allow investigating different processes related to renormalization of physical parameters in greater detail. So, in the model of heavy nucleon such concepts of virtual particle cloud and renormalization of mass as the result of nucleon's interaction with the vacuum become substantiated. At the same time, even slight increase in the complexity of the problem, such as increasing the number of the sorts of particle or considering finite mass of the nucleon, makes its analytical solution impossible. In this case approximate solution schemes should be used.

In work [9], variaitonal method was employed in order to obtain a nonperturbative solution of the scalar Yukawa model. By taking advantage of the simplicity of scalar fields and absence of divergent renormalization effects in two-dimensional space-time, this model allowed understanding the general strategy for solving similar problems without using the theory of perturbations.

In the following, we will consider a simplified model of nuclear (meson) forces, in which the nucleons and the mesons both are scalar particles, and the interaction between them is in Yukawa form $\lambda \phi^2 \chi$. From this point the problem formulation can be carried out in the space of any dimension.

Consider the problem of several particles (further – nucleons N) that interact via exchange of exchange bosons (further – mesons π) and that also interact with the vacuum via the processes of creation and destruction of virtual π-particles. Our model, in this way, describes the interaction of the sort $N \leftrightarrow N + \pi$ and is a scalar model of nuclear forces. The process $N \to N + \pi \to N$ (interaction of nucleon with the vacuum) is responsible for the renormalization of mass of the nucleons, and the process $N + N' \to N + \pi + N' \to N + N'$ (single-meson exchange) describes nucleon-nucleon interaction. The form of the processes that are described by such model demands a particular terms in the Hamiltonian b^+ba and bb^+a^+, which in turn defines the fundamental form of the interaction in the model. This model is analogous to the model of heavy nucleon with that difference that we take into account the finiteness of the nucleon mass. Thus, the Lagrangian becomes:

$$L = \frac{1}{2}\left(\partial_\mu \phi \cdot \partial^\mu \phi - {M_0}^2 \phi^2\right) + \frac{1}{2}\left(\partial_\mu \chi \cdot \partial^\mu \chi - m^2 \chi^2\right) + \lambda \phi^2 \cdot \chi \qquad (18)$$

Let us assume, nonetheless, that already at the stage of problem formulation one of the fields has preferred status – in particular the particles of the field ϕ (N) are conserved in the interaction but particles χ (π) can be both born and destroyed. Such asymmetry is reflected in the secondary-quantized Hamitonian (20), at the same time this asymmetry is not present in the Lagrangian (18). The Lagrangian, in particular, contains terms that only create or only destroy nucleons (creation and annihilation of nucleon-antinucleon pairs). This, generally speaking, requires one to consider an infinite chain of coupled integral equations in the Fock space for states with different number of nucleons. Mathematical solution of this situation consists in truncating the chain at a certain step, which corresponds variationally to considering the variational problem on a certain subset of states in the Fock space (for example, such that contain at most two nucleons). From physical point of view, such approach can be justified if the corrections from virtual pairs creation and annihilation is small, or if such contributions are already accounted for via renormalization of the coupling constant λ and meson's mass m.

Without delving too deep into discussion of this problem or logical consistency of truncation approach, we will consider that transition to considering variational anzats with fixed number of nucleons is justified after taking into account renormalization of the model's parameters due to the processes of creating and annihilation of virtual nucleons in the mass of the particles of field χ – m – and the coupling constant λ. Therefore, the terms of the Hamiltonian that have different number of creating and annihilation operators will not give a contribution into the variational approach since their averages will be calculated only on the states with fixed number of nucleons.

In order to construct the secondary-quantized Hamiltonian we perform transition to moment-space, which in equation (18) is performed using Fourier transform of the form:

$$\begin{aligned}\phi(x) &= \int d\vec{p}\,\frac{1}{\sqrt{(2\pi)^d\,2\omega(p,M)}}\left(e^{-i\vec{p}\vec{x}}b_p + e^{i\vec{p}\vec{x}}b_p^+\right)\\ \chi(x) &= \int d\vec{q}\,\frac{1}{\sqrt{(2\pi)^d\,2\omega(q,m)}}\left(e^{-i\vec{q}\vec{x}}a_q + e^{i\vec{q}\vec{x}}a_q^+\right)\end{aligned} \qquad (19)$$

where $\omega(q,m)=\sqrt{q^2+m^2}$, and the operators b^+, b and a^+, a are creation and annihilation operators for the particles of fields ϕ and χ, respectively, with the usual commutation relationships:

$$\left[b_q, b_p^+\right] = \delta^d(\vec{q}-\vec{p})$$

and analogous for operators a^+ and a, while all other commutators are zeroes.

In our model the renormalization of mass manifests itself in the change of the mass-surface condition $p^2 = -\vec{p}^2 + \omega^2 = M^2$, and finally in the form of the frequency-multiplier in the Fourier transformation of fields (19):

$$\phi(x) = \int dk\,\delta\left(k^2 - M^2\right)\cdot\left(\phi^+(x;k)b_p^+ + \phi^-(x;k)b_p^-\right) = \int d\vec{p}\,\frac{1}{\sqrt{\omega(p,M)}}\left(e^{-i\vec{p}\vec{x}}b_p + e^{i\vec{p}\vec{x}}b_p^+\right)$$

Use of relations (19) in original Lagrangian (18) gives:

$$\begin{aligned} &: H := \int d\vec{p}\cdot\Omega(p)b_p^+b_p + \int d\vec{q}\cdot\omega(q)a_q^+a_q - \\ &- g\int d\vec{p}\int d\vec{q}\,\frac{M}{\sqrt{\omega(p-q,m)\omega(p,M)\omega(q,M)}}b_p^+b_q\left(a_{q-p}^+ + a_{p-q}\right) \end{aligned} \qquad (20)$$

where the terms that contain b^+b and bb are discarded according to the discussion above. Also, for convenience, we defined $g = \frac{2\pi\cdot\lambda}{2\sqrt{2}M(2\pi)^{d/2}}$. For physically interesting case ($\lambda \approx 10 \div 20; M \approx 7$) in two dimensions $g \approx 0.5 \div 1$. Due to the change in the mass-surface condition, kinetic energy of free nucleon also changes as

$$\Omega(p) = \frac{\omega^2(p,M) + p^2 + {M_0}^2}{2\omega(p,M)} = \frac{2\omega^2(p,M) + {M_0}^2 - M^2}{2\omega(p,M)} = \omega(p,M) + \frac{{M_0}^2 - M^2}{2\omega(p,M)}. \qquad (21)$$

It can be easily seen that Hamiltonian (20), as in the case of heavy nucleon, the total number of nucleons and the total momentum is conserved. Therefore, we can assume subsequently that we only consider states with fixed number of nucleons and separated spatial motion (the total moment of the system is zero).

Mass M now becomes a quantity that can change depending on the parameters of the original Lagrangian. Identifying M with the physical mass of the nucleons, we can require from the variational principle that this parameter assumes the lowest possible value, which in turn corresponds to the lowest energy of the single-nucleon state.

Our problem will be to formulate the equation for the interaction of a nucleon with the vacuum as well as for the interactions of two nucleons, without restricting ourselves to the first several orders of the theory of perturbations. Symbolically, we need to formulate the problem for the set of variational states of the form:

$$|N=1\rangle = |N\rangle + |N\pi\rangle + |N\pi\pi\rangle + \ldots \text{ vacuum interactions} \qquad (22)$$

$$|N=2\rangle = |NN\rangle + |NN\pi\rangle + |NN\pi\pi\rangle + \ldots \text{ nucleon-nucleon interactions.} \qquad (23)$$

Considering that the mode l Hamiltonian (20) does not contain a direct interaction among the mesons, and using the analogy with the model of heavy nucleon, we will assume that

the distribution of mesons in all orders can be approximately factorized, and is thus defined by a single parameter – the distribution of such single mesons in momentum space. Thus, the coefficients of the variational state (6) for $a^+_{q_1}a^+_{q_2}\ldots a^+_{q_n}$ have to have the form,

$$c^{(1,n)}_{p;q_1q_2\ldots q_n} = F(q_1)F(q_2)\ldots F(q_n)f(p).$$

Then, using the form for the variational anzats (22), the state $|N=1\rangle$ has the form:

$$|\psi_1\rangle = \int d\vec{p}\sum_{n=0}^{\infty}\frac{g^n}{n!}\left(\prod_{i=1}^{n}\int d\vec{q}_i F(q_i)\right)\cdot f\left(p+\sum_{i=1}^{n}q_i\right)b_p^+\prod_{i=1}^{n}a^+_{q_i}|0\rangle \qquad (24)$$

where the integration is performed over d-dimensional volume, $F(p)$ and $f(p)$ - are variational functional parameters – in fact two form-factors for momentum-distribution of mesons and nucleons; $|0\rangle$ is the usual vacuum state of the Fock space. In this case, the translational motion of the system is reduced out and the total momentum of the system is zero.

The above anzats only contains one creation operator for nucleon and arbitrary number of creation operators for mesons, which corresponds, respectively, to the nucleon dressed in a cloud of virtual mesons – the state $|N=1\rangle$ symbolically defined above. The same can be written in a more compact form as:

$$\begin{aligned}
&|\psi_1\rangle = \int\frac{d\vec{x}}{(2\pi)^d}f(x)e^{gA^+(x)}\psi^+(x)|0\rangle;\\
&f(x)=\int d\vec{q}\cdot e^{-i\vec{q}\vec{x}}f(q),\psi^+(x)=\int d\vec{p}\cdot e^{i\vec{p}\vec{x}}b_p^+;\\
&A^+(x)=\int d\vec{q}\cdot F(q)e^{i\vec{q}\vec{x}}a_q^+
\end{aligned} \qquad (25)$$

Use of this anzats in the variational equation for mass-energy of the nucleon:

$$M=\min_{F,f}\frac{\langle\psi_1|:H:|\psi_1\rangle}{\langle\psi_1|\psi_1\rangle} \qquad (26)$$

produces an extremum problem for the functional:

$$M=\min_{F}\left[g^2\int d\vec{q}\,\omega(q)F^2(q)+\tilde{\Omega}(0)-2g^2\int d\vec{q}\frac{F(q)}{\sqrt{\omega(q)}}\tilde{H}(0,q)\right] \qquad (27)$$

The minimization with respect to parameter $f(p)^2$ can be carried out analytically and $f(p)^2=\delta^d(\vec{p})$, the mass of the mesons *m* can be reduced (chosen as one so that the mass of the nucleons is now in terms of the meson mass: $M\to M/m$), and respective quantities take on the form

$$\tilde{\Omega}(p)=\int d\vec{q}\int\frac{d\vec{x}}{(2\pi)^d}\Omega(q)e^{i(\vec{p}+\vec{q})\vec{x}}e^{g^2(G(x)-G_0)}=\int d\vec{q}\cdot\Omega(q)\left(e^{-g^2G_0}\delta(\vec{q}+\vec{p})+I(\vec{p}+\vec{q})\right);$$

$$\tilde{H}(p,q)=\int d\vec{q}'\int\frac{d\vec{x}}{(2\pi)^d}H(\vec{q}',\vec{q})e^{i(\vec{p}+\vec{q}')\vec{x}}e^{g^2(G(x)-G_0)}=$$

$$=\int d\vec{q}'\cdot H(\vec{q}',\vec{q})\left(e^{-g^2G_0}\delta(\vec{q}'+\vec{p})+I(\vec{p}+\vec{q}')\right);$$

$$I(\vec{p})=\int\frac{d\vec{x}}{(2\pi)^d}e^{i\vec{p}\vec{x}}\left(e^{g^2G(x)}-1\right)e^{-g^2G_0}$$

$$G(\vec{x})=\int d\vec{q}F^2(x)e^{-i\vec{q}\vec{x}};G_0=G(0)=\int d\vec{q}F^2(q)$$

$$H(\vec{p},\vec{q})=\frac{M}{\sqrt{\omega(\vec{p},M)\omega(\vec{p}-\vec{q},M)}}$$

(28)

where we also used the notation $\omega(q)=\omega_q=\omega(q,1)=\sqrt{q^2+1}$. As we will see further, the quantity $\tilde{\Omega}$ has the meaning of kinetic energy of the nucleon altered due to "dressing" of the nucleon with virtual mesons, and $\tilde{H}$ - is the modified interaction potential.

Obtained relationships completely define the state of a single nucleon interacting with the vacuum, and allow to obtain the energy of this state (physical mass) for any values of the parameters of the original Lagrangian – the coupling constant *g* and bare mass M_0. In fact, ultraviolet divergences aside, considering the renormalization of nucleon's mass in this model does not present any difficulties.

In order to obtain the equation for two interacting nucleons we have to consider the anzats (23). Similar to the above case, we can consider such a state as a superposition of two "dressed" nucleons with certain form-factor $\psi(p)$:

$$|\psi_2\rangle=\frac{1}{\sqrt{2}}\int dp\cdot\psi(p)\tilde{b}_p^+\tilde{b}_{-p}^+|0\rangle$$

Where the annihilation operator of "dressed" nucleons is:

$$\tilde{b}_x=b_xe^{A(x)}$$

Although at this stage the form-factor *F(q)* needs to be considered as the variational parameter such as the function $\psi(p)$, we can also think that for *F(q)* we can use the solution obtained from singe-nucleon problem. Then, together with the definition of the physical nucleon mass (27), we obtain:

$$\left(2\left(\tilde{\Omega}(p)-\tilde{\Omega}(0)+M\right)-M_2\right)\Psi(p)=2g^2\int d\vec{q}\left(2\frac{F(q-p)}{\sqrt{\omega_{q-p}}}\tilde{H}(p,q-p)-\omega_{q-p}F^2(q-p)\right)\Psi(q) \quad (29)$$

Here, the mass *M* and the form-factor *F(q)* – are the quantities corresponding to the solutions of the single-nucleon problem. As we can see, the single-nucleon state is therefore a necessary step towards construction of the effective two-nucleon wave-equation. Also let us note that because of taking into account the cloud of virtual meson surrounding nucleon we have not only the effective potential for nucleon-nucleon interaction, but also a modification of the kinetic energy of the nucleons, and the coupling constant *g* enters not only the effective potential but also the kinetic energy.

The relationships we obtain above are the integral equations that describe both the problem of two-nucleon scattering as well as two-nucleon bound state, which is one of its important advantages. At the same time, numerical solution of that equation is nontrivial. In [9], this equation is considered numerically in one dimension using the simples model functions $F(p)$. In two dimensions the problem of solving these integral equations becomes significantly more complicated due to increase in the dimension of associated integrals.

4. Mass renormalization in two-dimensional scalar Yukawa model

In the previous section we formulated the problem for analysis of single- and two-nucleon states in scalar Yukawa model in any dimension. Accordingly, the problem of the renormalization of nucleon mass becomes a problem of finding the extremum of (26): $\frac{\langle\Phi_1(F)|:H:|\Phi_1(F)\rangle}{\langle\Phi_1(F)|\Phi_1(F)\rangle}$, where $:H:$ and $|\Phi_1(F)\rangle$ are normally ordered Hamiltonian (20) and the state (25), respectively. The variation is performed with respect to the functional parameter $F(q)$. Thus formulated problem allows one to numerically calculate physical nucleon mass as the function of the coupling constant g and bare mass M_0, in all perturbation of the perturbations theory, due to specific form of the variational anzats. In practice, the more interesting problem is finding of the "bare" mass via the physical mass, or solving the renormalization problem using the known physical mass and coupling constant. In [22], for example, the analysis is performed in exactly that form. In particular, by solving equations (21) and (27) we obtain for the bare mass:

$$M_0 = \sqrt{M^2 + 2Mg^2C(g)} \tag{30}$$

where the functional $C(g)$ is defined in the form: $C(g) = \max_F \left\{ f_1 - \frac{f_2 + f_3 + e_1 - e_2}{e_0} \right\}$ (31)

with following notation:

$$\begin{aligned}
f_1 &= 2\int d\vec{q}\,\frac{F(q)}{\sqrt{\omega_q}};\\
f_2 &= 2\int d\vec{q}\,\frac{F(q)}{\sqrt{\omega_q}}\left(1-\sqrt{\frac{M}{\omega(q,M)}}\right);\\
f_3 &= \int d\vec{q}\cdot\omega_q F^2(q);\\
e_0 &= 1-\int d\vec{q}\cdot I(\vec{q})\left(1-\frac{M}{\omega(q,M)}\right);\\
e_1 &= \int d\vec{q}\cdot I(\vec{q})\frac{\omega(q,M)-M}{g^2};\\
e_2 &= 2\int d\vec{q}\,\frac{F(q)}{\sqrt{\omega_q}}\int d\vec{p}\cdot I(\vec{p})\{H(0,0)-H(\vec{p},0)-H(0,\vec{q})+H(\vec{p},\vec{q})\}
\end{aligned} \tag{32}$$

where $H(\vec{p},\vec{q}), I(\vec{q}), G(\vec{x})$ are defined in (28). The coupling constant g now only enters into the quantities e_0, e_1, e_2, which represent the interaction, and does not appear in f_1, f_2, f_3, which represent kinetic energy. Partition (31) is due to its convenience for analytical analy-

sis of limiting cases for g and numerical convenience, and corresponds to partition of the functional according to the terms of different order as $g \to 0$. In particular, the most complex term – e_2 – has the order of g^2. Essentially, the variational parameter enters the functional via the terms e_0, e_1, e_2 in a complex – exponential – form. Thus, analytical solution of this problem in general case is impossible.

At the same time, in the case of small or large g, small or large $M := \frac{M}{m}$, it is possible to solve this problem analytically.

4.1 The limit of weak and strong coupling

One of the simplest analytical approximations is the approximation nof weak coupling $g \to 0$ that allows to consider a problem using regular perturbation theory [9,22]. In this case the most complex component of representation (28) – the integral $I(q)$ – can be simplified in this way:

$$\begin{aligned} I(\vec{q}) &= \int \frac{d\vec{x}}{(2\pi)^d} e^{i\vec{q}\vec{x}} e^{-g^2 G_0} \left(e^{g^2 G(\vec{x})} - 1 \right) = \int \frac{d\vec{x}}{(2\pi)^d} e^{i\vec{q}\vec{x}} g^2 G(\vec{x}) + O\left(g^4\right) = \\ &= \int \frac{d\vec{x} d\vec{p}}{(2\pi)^d} g^2 F^2(\vec{p}) e^{-i(\vec{p}-\vec{q})\vec{x}} + O\left(g^4\right) = g^2 F^2(\vec{q}) + O\left(g^4\right) \end{aligned} \tag{33}$$

Thus, having $I(q) \approx g^2 F^2(q)$, we obtain $e_2 \approx O\left(g^2\right)$, $e_0 \approx 1 - O\left(g^2\right)$ and $e_1 \approx \int d\vec{q} F^2(\vec{q})(\omega(q,M) - M) + O\left(g^2\right)$. Using these equations in the functional *C[F]*, we obtain:

$$C(F;g) = 2\int d\vec{q} \left\{ \frac{F(\vec{q})}{\sqrt{\omega_q}} \sqrt{\frac{M}{\omega(q,M)}} - \frac{F^2(\vec{q})}{2} \left(\omega_q + \omega(q,M) - M\right) \right\} + O\left(g^2\right). \tag{34}$$

It is easy to perform variation in (34) with respect to $F(q)$ and obtain for the extremum, up to $O(g^2)$:

$$F(\vec{q}) = \sqrt{\frac{M}{\omega_q \omega(q,M)}} \frac{1}{\omega_q + \omega(q,M) - M}, \tag{35}$$

and for the extremal value of the functional:

$$C\left(g^2\right) = M \int \frac{d\vec{q}}{\omega_q \omega(q,M) \left(\omega_q + \omega(q,M) - M\right)} + O\left(g^2\right) \tag{36}$$

We can see here that, formally, the limit of weak coupling can be taken without regard to the space's dimensionality. Equation (36) also can be considered to be correct for any dimensionality. In particular, its analysis for D=1,2,3 dimensional space in the limit of heavy nucleon $M \to \infty$ gives $C(0) \approx \pi - \frac{1}{M}\ln\left(\frac{M}{C}\right), M \to \infty$ for D=1, $C(0) \approx cg^2 \ln M$ for D=2 and $C(0) \approx cg^2 M \ln\left(\frac{\Lambda}{M}\right)$ for D=3 [9]. On the other hand, for D=3 this result should be considered after removing ultraviolet divergences. Divergence of (36) in D=3 can be easily seen if we consider that the integrand's denominator grows as q^3 while numerator grows as q^2 (as q tends to infinity), thus the entire integral diverges logarithmically as the function of the upper limit of the integration.

From rigorous mathematical point of view that means that the functional *C(g; F)* does not have a regular maximum on the set of our variational, and in particular can achieve anyhow large values regarding the parameter *F(q)*. Since $C(g) = \frac{{M_0}^2 - M^2}{2g^2 M}$, for any finite bare mass M_0 the relevant physical mass, therefore, will tend to zero and vice-verse – any finite physical mass causes an infinite bare mass. This is a typical situation in quantum field theories normally called the problem of ultraviolet divergences. In D=1 and D=2 this problem does not appear.

The case $g \to 0$ in D=1 is also considered in [22]. In particular, in that work the produced limit is investigated in the case of light nucleons, $M \to 0$, where it is shown that $C(0) \approx 2M \ln\left(\frac{C}{M}\right)$. In the same work it is shown that the first term in the expansion of *C(g)* for small masses coincides with the regular theory of perturbations.

The case of D=2 is the only case where the integral (36) can be calculated analytically. For that let us write that integral in the form:

$$C(0) = \pi \cdot M \int_0^\infty \frac{dz}{\sqrt{z+1}\sqrt{z+M^2}\left(\sqrt{z+1}+\sqrt{z+M^2}-M\right)},$$

where we performed the integration over the angles with the account for cylindrical symmetry and performed a substitution $qdq \to d\frac{q^2}{2} \to \frac{dz}{2}$. Using the substitution:

$$z = \frac{1}{4}\left(\frac{M^2}{M^2-1}t^2 + \frac{M^2-1}{M^2 t^2} + 2\,\mathrm{sgn}\left(M^2-1\right)\right)\left(M^2-1\right) - 1$$

The integral reduced to $C(0) = M\pi \int_{\frac{M+1}{M}}^{\infty} \frac{2}{M} \frac{dt}{t(t+1)} = 2\pi \ln(M+1)$. (37) Як бачи

It is also useful to consider the extremal form-factor in the limits of small and large nucleon mass. In these limits it is easy to obtain following results:

$$F(q) \approx \sqrt{\frac{1}{\omega_q}} \frac{1}{\omega_q + \frac{1}{2M} q^2}, M \to \infty$$
$$F(q) \approx \sqrt{\frac{M}{q \cdot \omega_q}} \frac{1}{\omega_q + M(q-1)}, M \to 0 \qquad (38)$$

Last equation, in particular, cannot be used in D=1 and normally causes divergence there; thus, this limit formally remaining correct in D=1 can be only used in D=2.

It is important that in the framework of variaitonal approach the renormalization (30-32) allows to consider also the model in the limit of strong coupling $g \to \infty$, without any reformulation of the theory. In particular, in D=1 one can obtain [22]:

$$C(g) \approx \sqrt{\frac{M}{g}} \ln\left(\frac{g}{M}\right);$$
$$F(q) \approx \frac{a}{\sqrt{\omega_q}\left(\omega_q + \frac{\alpha}{2} q^2\right)}. \qquad (39)$$

In this thesis we show that formal results in this approximation can be obtain for any dimensionality of space (naturally, up to the problem of ultraviolet divergences).

The case of strong coupling can be considered using stationary phase approximation. Using that in *I(q)* the constant *g* enters under exponent, according to stationary phase approximation we expand *G(x)* into series in *x* and only keep the leading term,

$$I(q) = \frac{1}{(2\pi)^d} \int d\vec{x} e^{i\vec{q}\vec{x}} e^{-g^2 G_0} \left(e^{g^2 G(x)} - 1\right) \approx e^{-g^2 G_0} \delta(\vec{q}) + \int \frac{d\vec{x}}{(2\pi)^d} e^{i\vec{q}\vec{x}} e^{g^2 \Delta G(x)} \approx$$
$$\approx \int \frac{d\vec{x}}{(2\pi)^d} e^{i\vec{q}\vec{x}} e^{-g^2 A^2 x^2} = \frac{1}{\pi^{d/2}} \frac{1}{(2gA)^d} e^{-\frac{q^2}{4g^2 A^2}} = \frac{1}{\pi^{d/2}} \frac{1}{\bar{g}^d} e^{-\frac{q^2}{\bar{g}^2}}, \qquad (40)$$

where we used the fact that e_0, e_1, e_2 are defined in such a way that adding and subtracting δ-functions does not change them, thus,

$$\Delta G(x) = \int d\vec{p} F^2(p)\left(e^{-i\vec{p}\vec{x}} - 1\right) = \int d\vec{p} F^2(p)\left(-i\vec{p}\vec{x} - (\vec{p}\vec{x})^2 + \ldots\right) \approx x^2 \int p^2 F^2(p) d\vec{p}, \qquad (41)$$

where only the leading term is kept (the terms proportional to *x* gives zero as the result of spherical symmetry). For the terms of (31), then,

$$e_0 = 1 - \int d\vec{q} I(\vec{q}) \left(1 - \frac{M}{\omega(q,M)}\right) \approx$$

$$\approx 1 - \int d\vec{q} \frac{1}{\pi^{d/2}} \frac{1}{\bar{g}^d} e^{-\frac{q^2}{\bar{g}^2}} + \int \frac{d\vec{q}}{\pi^{d/2}\bar{g}^d} \frac{M}{\omega(q,M)} e^{-\frac{q^2}{\bar{g}^2}} = , \qquad (42)$$

$$= \int \frac{d\vec{z}}{\pi^{d/2}} e^{-z^2} \frac{1}{\omega\left(\frac{\bar{g}}{M} z\right)}$$

where $\bar{g} = 2gA$ – in some sense an effective coupling constant, and $z = q/\bar{g}$. In the limit of large $\bar{g}$ the function $\omega\left(\frac{z}{\Lambda}\right) = \omega\left(\frac{\bar{g}}{M} z\right) = \omega\left(\frac{2gA}{M} z\right)$ can be replaced with $\frac{z}{\Lambda}$ $(\Lambda \to 0, g \to \infty)$ giving:

$$e_0 \approx \Lambda \Omega_d \frac{\Gamma\left(\frac{d-1}{2}\right)}{2\pi^{d/2}}, \qquad (43)$$

where Ω_d is d-dimensional solid angle. Similarly for e_1:

$$e_1 = \int d\vec{q} I(\vec{q}) \left(\frac{\omega(q,M) - M}{g^2}\right) \approx \frac{M}{\pi^{d/2} g^2} \int d\vec{z} e^{-z^2} \left(\omega(z/\Lambda) - 1\right) \approx \frac{M}{\Lambda g^2} \frac{\Gamma\left(\frac{d+1}{2}\right)}{2\pi^{d/2}}; \qquad (44)$$

$$e_1 = O(A/g)$$

Thus, e_1 can be neglected in enumerator relative to $f_3 = O(1)$. Finally for e_2 we have:

$$f_2 - e_2 \approx \int d\vec{p} \frac{F(p)}{\sqrt{\omega_p}} \int \frac{d\vec{q}}{\pi^{d/2}\bar{g}^d} e^{-\frac{q^2}{\bar{g}^2}} \left(\frac{M}{\sqrt{q^2 + M^2}} - \frac{M}{\sqrt[4]{\left(q^2 + M^2\right)\left(|\vec{q} - \vec{p}|^2 + M^2\right)}}\right) =$$

$$= \int \vec{p} \frac{F(p)}{\sqrt{\omega_p}} \int \frac{dz}{\pi^{d/2}} e^{-z^2} \left(\frac{M}{\sqrt{z/\Lambda + M^2}} - \frac{M}{\sqrt[4]{\left(z/\Lambda + M^2\right)\left(|\vec{z}/\Lambda - \vec{p}|^2 + M^2\right)}}\right) = O\left(\Lambda^2\right) \qquad (45)$$

Note that nowhere we specifically used the dimensionality of space here, with the exception for e_0, e_1 where we have $\Gamma\left(\frac{d-1}{2}\right), \Gamma\left(\frac{d+1}{2}\right)$. This makes the above calculation formally correct for all dimensions. Then the functional *C(g)* is defined by the relation:

$$C(F;g) = f_1 - \frac{f_3}{e_0}, \qquad (46)$$

and the extremal form-factor is:

$$F(q) = \frac{a}{\sqrt{\omega_q}\left(\omega_q + \frac{\alpha}{2} q^2\right)} \qquad (47)$$

The form-factor (47) coincides with (39), that is the case D=1 is included in the above calculation. Solution of the extremum problem in parametric form leads to these results. Parameter *a* enters *C(g)* quadratically and can be excluded analytically,

$$a^*(\alpha)=\sqrt{\frac{f_1(\alpha)}{6 f_3(\alpha)A(\alpha)}\frac{C_d M}{g}}$$
$$C^*(\alpha)=\frac{2}{3} f_1(\alpha)\sqrt{\frac{C_d f_1(\alpha)}{f_3(\alpha)A(\alpha)}}\sqrt{\frac{M}{g}}, \quad (48)$$

where f_1, f_3, A are certain functions only of α and the dimensionality of space, and $C_d = \frac{\Gamma((d-1)/2)}{2\pi^{d/2}}\Omega_d$ - a constant. Importantly, all external parameters (mass *M*, coupling *g*) enter this formula as factors. This allows to re-write (48) as $C(g;\alpha)=C_d^*(\alpha)\sqrt{\frac{M}{g}}$ and find the maximum in α independent of such parameters. This latter problem has to solved numerically, and in particular in D=3 we also face the problem of ultraviolet divergences. However, formally we still can consider that $C(g)\approx C_d\sqrt{M/g}$. Using this together with (39), we finally have:

$$C(g)\approx C_1\sqrt{\frac{M}{g}}\ln\left(\frac{g}{M}\right), N=1$$
$$C(g)\approx C_d\sqrt{\frac{M}{g}}, N\geq 2 \quad (49)$$

Additionally the parameter *A* de[ends on the mass and the coupling as:

$$A\approx\sqrt{M/g}, g\to\infty \quad (50).$$

For D=2, C_2 can be explicitly calculated as $C_2\approx 2.566\ldots$; $\alpha_2 = 0.1604\ldots$

It should be also noted that while the case D=3 formally is covered by the above calculations, in reality should be considered together with the proper renormalization of the theory.

4.2 The limit of light and heavy nucleous

Other interesting limiting cases that can be considered analytically are the limits of light and heavy nucleons. Consider $M\to\infty$. The most complex terms *e*, which contain exponential dependencies on form-factor and the coupling constant can be simplified in this case. In particular:

$$e_0 = 1 - \int I(q)\left(1 - \frac{M}{\omega(q,M)}\right)d\vec{q} \approx 1 - \frac{1}{2M^2}\int I(q)q^2 dq + O\left(\frac{1}{M^4}\right);$$
$$e_1 = \int d\vec{q} I(q)\frac{\omega(q,M) - M}{g^2} \approx \frac{1}{2Mg^2}\int d\vec{q} I(q)q^2 + O\left(\frac{1}{M^3}\right); \qquad (51)$$
$$e_2 \approx \int d\vec{p}\,\frac{F(p)}{\sqrt{\omega_q}}\int d\vec{q}\cdot f(p,q)\frac{\vec{q}\cdot\vec{p}}{M^2} + O\left(\frac{1}{M^4}\right) = O\left(\frac{1}{M^4}\right)$$

Although we can expect that the term e_1 can be neglected next to f_3, the calculation shows that it is not the case. Thus, we will keep this term for now, writing

$$C(F;g) = f_1 - \frac{f_3 + f_2 + e_1}{e_0}. \qquad (52)$$

Furthermore we use the following fact that:

$$\int d\vec{q} I(q)q^2 = \sum_i \int \frac{d\vec{q}d\vec{x}}{(2\pi)^d}\cdot q_i{}^2 e^{i\vec{q}\vec{x}} e^{g^2\Delta G(x)} = \sum_i \int d\vec{x}\cdot e^{g^2\Delta G(x)}\left(-\frac{\partial^2}{\partial x_i{}^2}\delta(x_i)\right) =$$
$$= \sum_i \left(-\frac{\partial^2}{\partial x_i{}^2} e^{g^2\Delta G(x)}\right)\Bigg|_{x=0} = g^2\int d\vec{p} p^2 F^2(p) = g^2 A^2$$

thus the solution of the extremum problem for (52) gives:

$$F(q) \approx \frac{1}{\sqrt{\omega_q\omega_{q/M}}}\frac{1}{\omega_q + \left(1 + d\,\frac{g^2}{M}\right)\frac{q^2}{2M}}\left(1 - \frac{g^2A^2}{2M^2}\sqrt{\omega_{q/M}}\right) \approx$$
$$\approx \frac{1}{\sqrt{\omega_q\omega_{q/M}}}\frac{1}{\omega_q + \frac{q^2}{2M}} \approx \frac{1}{\sqrt{\omega_q}}\frac{1}{\omega_q + \frac{q^2}{2M}} \qquad (53)$$

We can see that in the first approximation (53) coincides with the form-factor of weak coupling for large masses (38), thus it is easy to obtain $C(g,M) \approx 2\pi\ln(M)$.

Another option for this limit is considering a simplified form-factor. Such a form-factor can be a function in the form (47), which, from one side, is a limiting case of the first term in (53) for large nucleon masses and, from another side, has similar behavior with the second term in (53) for large moments. Solution of the variational problem in this case in parametric form gives:

$$C(M;g) \approx 2\pi\ln(2xM) + 2\pi(1 - x(4C_1 + 1)) - \frac{4\pi^2 g^2 x}{M}\ln(2xM) + \ldots, \qquad (54)$$

where $x \approx 15.55; C_1 \approx 0.06873$ are certain constant, and parameters are defined as:

$$\alpha = \frac{C_1}{1+C_1}\frac{1}{M} = \frac{1}{xM};$$
$$a \approx \frac{\ln 2xM - 2xC_1}{\ln 2xM + x - 1} + O\left(\frac{1}{M}\right) \approx 1, M \to \infty \quad (55)$$

Obtained results correctly reflects the dependence of *C(g,M)* from mass and coupling.

For the case of small masses *M* let us notice that the form-factor has similar factor $\sqrt{M}$ in all limits considered so far – weak coupling limit, strong coupling limity (see Eq. (50)), and large mass. Therefore we can assume here that such factor can multiply the form-factor also in this case.

Thus, we represent the form-factor in a more general case: $F(q,M) = M^a f(q,M)$, where 0<*a*<1 and function *f(q,M)* has order of one when $M \to 0$. In this case, $G(x) \to 0$, $M \to 0$, and formally the calculation resembles that for the weak coupling limit. Thus:

$$I(q) \approx M^{2a} g^2 f^2(q;M);$$
$$e_0 \approx 1 - g^2 M^{2a} \int f^2(q;M)\left(1 - \frac{M}{\omega(q,M)}\right)d\vec{q} \approx 1;$$
$$e_1 \approx M^{2a} g^2 \int f^2(q;M)\frac{\omega(q,M)-1}{g^2}d\vec{q};$$
$$e_2 \approx M^{3a} \approx 0$$

from where we can establish the necessary similarities. Setting $a = 1/2$, we can exclude the factor from the mass to obtain:

$$F(q;M) = \frac{\sqrt{M}}{\sqrt{q\omega_q}}\frac{1}{\omega_q + M(q-1)};$$
$$C(M) \approx 2\pi M, M \to 0 \quad (56)$$

Interestingly, both the approximations of light and heavy nucleons can be obtained from the form-factor of weak coupling approximation (38). In particular, this suggests that for any coupling *g* the form-factor, in fact, is determined by its asymptotic behavior $g \to 0$.

Finally, let us list some of the main properties of the form-factor for the limits considered here, that will be useful for properly selecting the class of functions on which to solve the general variational problem:

All form-factors are spherically symmetric;

All form-factors have the form $\frac{1}{\sqrt{\omega_q \omega_{q/M}}}\frac{a + b\omega_{q/M}}{\omega_q + \frac{\alpha}{2}(\omega_{q/M} - 1)}$ or $\frac{1}{\sqrt{\omega_q \omega_{q/M}}}\frac{a + b\omega_{q/M}}{\omega_q + \frac{\alpha}{2}q^2}$

;

All form-factors contain the factor $\sqrt{M}$.

5. Numerical solution of the renormalization problem

For arbitrary mass *M* and coupling constant *g* the functional relationships (30-32) required numerical solution. A simplest calculation of this sort had been performed in [9], where the form-factor was used in the form $F(q) = a / \omega_q^{3/2}$ and a good lower bound on the bare mass M_0 had been produced. In [22], variational approach had been applied on a class of variational functions defined as superpositions of certain base-functions, which allows to solve the problem with controlled accuracy. In this work we similarly use this latter method in two-dimensional case.

Thus, for calculations of the dependency $M_0(M, g)$ (30) we could use a numerical variational approach using the class of test functions *F(q)* in the form of superposition of Yukawa terms:

$$F(q; g, M) = \sum_{i=1}^{N} \frac{a_i}{g\left(1 + (\alpha_i q)^2\right)}. \qquad (57)$$

The general properties of the form-factor listed above, however, lead us to conclude that a better approximation could be:

$$F(q; g, M) = \sum_{i=1}^{N} \frac{a_i}{\sqrt{\omega_q}} \frac{1}{\omega_q + \frac{\alpha_i}{2} q^2}\ , \qquad (58)$$

that has appropriate behavior for strong coupling, large mass, as well as weak coupling, and at the same time does not contain too many variational parameters. From the other hand, analytical construction of functions *G(x)* in this representation is not possible, which complicates the problem of numerical calculation of *C[F]*. Also, form-factor (57) correctly mimics the behavior of (58) for small distances, is spherically symmetric, and has a small number of variational parameters. Furthermore, most of the components of *C[F]* can be calculated analytically. Investigation of the quality of the form-factor in the form $F(q) = \frac{a}{\left(1 + (qb)^2\right)^{(n+3)/4}}$ had shown that the best results are obtained also when *n*=1. These allow us to assume that the representation (57) will be a suitable choice for the class of variational functions.

Representation of the form-factor as a superposition of certain basis functions also allows to produce results with controlled precision. In practice, the above choice results in

very fast convergence of the estimates with single component producing close to 5% accuracy.

One of the most important properties of the representation (57) is that almost all quantities in *C[F]* can be evaluated analytically. In particular for f_1 and f_3 we have

$$\begin{aligned} f_1 &= 2\int d\vec{q}\,\frac{F(q)}{\sqrt{\omega_q}} = 2\cdot\frac{2\pi}{g}\sum_i a_i \int_0^\infty \frac{qdq}{\left(1+(\alpha_i q)^2\right)\sqrt[4]{1+q^2}} = \\ &= \frac{2\pi}{g}\sum a_i \int_0^\infty \frac{dz}{\left(1+\alpha_i{}^2 z\right)\sqrt{1+z}} = \frac{8\pi}{g}\sum {}_2F_1\left(1,1;\frac{5}{4};1-\alpha_i{}^2\right)a_i \end{aligned}; \tag{59}$$

$$\begin{aligned} f_3 &= \int d\vec{q}\omega_q F^2(q) = \frac{2\pi}{g}\sum_{i<j}\frac{a_i a_j}{\alpha_i\alpha_j\left(\alpha_j{}^2-\alpha_i{}^2\right)}\left\{\begin{matrix}\alpha_j\sqrt{1-\alpha_i{}^2}\left(\pi-2\arcsin\alpha_i\right)-\\ -\alpha_i\sqrt{1-\alpha_j{}^2}\left(\pi-2\arcsin\alpha_j\right)\end{matrix}\right\}+ \\ &+\frac{\pi}{g}\sum_i a_i{}^2\,\frac{2\alpha_i\sqrt{1-\alpha_i{}^2}+\pi-2\arcsin\alpha_i}{2\alpha_i{}^3\sqrt{1-\alpha_i{}^2}}; \end{aligned} \tag{60}$$

and using Hankel's transformation [27],

$$\int_0^\infty \frac{J_0\left(\sqrt{z}\right)}{(z+a)^\alpha}dz = \frac{2^{2-\alpha}a^{\frac{1-\alpha}{2}}}{\Gamma(\alpha)}\mathrm{K}_{1-\alpha}\left(\sqrt{a}\right) \tag{61}$$

we finally obtain for *G(x)* (which is the most important given that this quantity enters many-dimensional integrals in e_2) from (28, 57, 61):

$$\begin{aligned} g^2G(x) &= \int d\vec{q}F^2(q)e^{-i\vec{q}\vec{x}} = 2\pi\int_0^\infty J_0(qx)F^2(q)dq = \\ &= 4\pi\sum_{i<j}a_i a_j\int_0^\infty\frac{J_0(qx)}{\left(1+(\alpha_i q)^2\right)\left(1+(\alpha_j q)^2\right)}qdq + 2\pi\sum_i a_i{}^2\int_0^\infty\frac{J_0(qx)}{\left(1+(\alpha_i q)^2\right)^2}qdq = \\ &= 4\pi\sum_{i<j}\frac{a_i a_j}{\alpha_i{}^2-\alpha_j{}^2}\left\{\mathrm{K}_0\left(\frac{x}{\alpha_i}\right)-\mathrm{K}_0\left(\frac{x}{\alpha_j}\right)\right\}+\pi\sum_i\frac{a_i{}^2}{\alpha_i{}^3}\mathrm{K}_1\left(\frac{x}{\alpha_i}\right); \\ g^2G_0 &= G(0) = 4\pi\sum_{i<j}\frac{a_i a_j}{\alpha_i{}^2-\alpha_j{}^2}\left\{\ln\alpha_i-\ln\alpha_j\right\}+\pi\sum_i\frac{a_i{}^2}{\alpha_i{}^2}; \end{aligned} \tag{62}$$

At the same time, the form-factor (58), while allowing to calculate analytically f_1, f_3, does not allow to calculate *G(x)*, and in that respect is much less suitable for numerical solution of this problem. The quantity f_2 cannot be obtained analytically in either representation, however, due to spherical symmetry, can be reduced to a 1D integral with good convergence properties. Calculation of this integral does not present problems. Finally, the integrals e_0 and e_1 can be simplified to 1D integrals as:

$$e_0 = 1 - \int d\vec{q} I(q)\left(1 - \frac{M}{\omega(q,M)}\right) = 1 - \int d\vec{q} I(q) + \int d\vec{q} I(q)\frac{M}{\omega(q,M)} =$$

$$= \int\int \frac{d\vec{q}d\vec{x}}{(2\pi)^d} e^{g^2\Delta G(x)} e^{i\vec{q}\vec{x}} \frac{M}{\omega(q,M)} = \int d\vec{x} e^{g^2\Delta G\left(\frac{x}{M}\right)} \int \frac{d\vec{q}}{(2\pi)^d} \frac{e^{i\vec{q}\vec{x}}}{\omega(q)} = \qquad (63)$$

$$\int d\vec{x} e^{g^2\Delta G\left(\frac{x}{M}\right)} F\left[\frac{1}{\omega(q)}\right](x)$$

where $F[f(q)](x)$ is the Fourier image of f in d-dimensional space. In the case of D=2, this function can be calculated using (61) and equals:

$$F\left[\frac{1}{\sqrt{q^2+1}}\right] = \int \frac{d\vec{q}}{(2\pi)^2} \frac{e^{i\vec{q}\vec{x}}}{\sqrt{q^2+1}} = \frac{1}{2\pi}\int_0^\infty \frac{J_0(qx)}{\sqrt{q^2+1}} qdq = \frac{e^{-x}}{2\pi x}. \qquad (64)$$

Finally: $e_0 = \int_0^\infty dx \cdot e^{g^2\Delta G\left(\frac{x}{M}\right) - x}$. Calculation of this integral with analytically defined *G(x)*is not difficult. Similarly for e_1:

$$e_1 = \int d\vec{q} I(q) \frac{\omega(q,M) - M}{g^2} = \int d\vec{x} e^{g^2\Delta G\left(\frac{x}{M}\right)} M \int \frac{d\vec{q}}{(2\pi)^d} e^{i\vec{q}\vec{x}} \frac{\omega(q)-1}{g^2} =$$

$$= \frac{M}{g^2}\int_0^\infty dx \frac{x+1}{x} e^{-x}\left(1 - e^{g^2\Delta G(x/M)}\right) \qquad (65)$$

where we took into account that $\int \frac{d\vec{q}}{(2\pi)^2} e^{i\vec{q}\vec{x}}\sqrt{q^2+1} = \frac{1}{2\pi}\int_0^\infty \frac{J_0(qx)}{\sqrt{q^2+1}} qdq = -\frac{x+1}{2\pi\cdot x^3} e^{-x}$ and the integral $\int d\vec{x} e^{g^2\Delta G(x)} \int \frac{d\vec{q}}{(2\pi)^d} \frac{M}{g^2} e^{i\vec{q}\vec{x}} = \frac{M}{g^2}$ is represented in the form

$$\frac{M}{g^2} = \int d\vec{x} \int \frac{d\vec{q}}{(2\pi)^d} e^{i\vec{q}\vec{x}} \frac{\sqrt{q^2+M^2}}{g^2}.$$

For e_2 the situation, however, is worse. In general, the term e_2 gives relatively small contribution to the mass, but only as long as g is below unity. For stronger coupling this term grows very quickly and becomes important already at $g \approx 1$, changing the resulting form-factor as well as M_0. In general, in D=2 e_2 contains a 6-dimensional integral, of which two can be calculated from spherical symmetry. One more integral over angular variables can be calculated analytically, and 3-dimensional integration over radial variables has to be performed numerically. This substantially complicates the situation as compared with D=1, where one has only two-dimensional integration with relatively simple integrated functions. At the same time, this situation is much simpler from the case of two-particle equation

which requires calculating 5-dimensional integral. At this time we do not have suitable algorithms to calculate integrals of such high dimensionality. Therefore, in order to calculate e_2 certain simplifications need to be made. We break this component into following parts $e_2 = e_{21} + e_{22} + e_{23} + e_{24}$, where

$$e_{21} = 2\int d\vec{p}\,\frac{F(p)}{\sqrt{\omega_p}}\int d\vec{q}\int \frac{d\vec{x}}{(2\pi)^d} e^{i\vec{q}\vec{x}}\left(e^{g^2\Delta G(x)} - e^{-g^2G_0}\right) g(0,0) = f_1\cdot\left(1 - e^{-g^2G_0}\right);$$

$$e_{22} = -2\int d\vec{p}\,\frac{F(p)}{\sqrt{\omega_p}}\int d\vec{q}\int \frac{d\vec{x}}{(2\pi)^d} e^{i\vec{q}\vec{x}}\left(e^{g^2\Delta G(x)} - e^{-g^2G_0}\right) g(q,0) = f_1\cdot\left(e^{-g^2G_0} - e_0\right);$$

$$e_{23} = -2\int d\vec{p}\,\frac{F(p)}{\sqrt{\omega_p}}\int d\vec{q}\int \frac{d\vec{x}}{(2\pi)^d} e^{i\vec{q}\vec{x}}\left(e^{g^2\Delta G(x)} - e^{-g^2G_0}\right) g(0,p) = \left(1 - e^{-g^2G_0}\right)\cdot(f_2 - f_1);$$

$$e_{24} = 2\int d\vec{p}\,\frac{F(p)}{\sqrt{\omega_p}}\int d\vec{q}\int \frac{d\vec{x}}{(2\pi)^d} e^{i\vec{q}\vec{x}}\left(e^{g^2\Delta G(x)} - e^{-g^2G_0}\right)\frac{M}{\sqrt[4]{\left(q^2+M^2\right)\left((\vec{q}-\vec{p})^2+M^2\right)}}$$

Analysis shows that the most suitable is merging e_{22} and e_{24}, as such containing a function that tends to zero with growing moment q:

$$\frac{M}{\sqrt[4]{\left(q^2+M^2\right)\left((\vec{q}-\vec{p})^2+M^2\right)}} - \frac{M}{\sqrt[4]{\left(q^2+M^2\right)\left(q^2+M^2\right)}}$$

Thus, calculation of e_2 is organized in this sequence: integration over angles $\vec{p},\vec{q},\vec{x}$ is done analytically according to the formulas below. Integration over q is performed the last (since the integrals over p and x can be performed separately as intersecting only over q).

Let us consider in more detail these steps. Represent e_{24} in the form:

$$e_{24} = 2\int d\vec{q}\,\frac{1}{\sqrt[4]{q^2+M^2}}\int \frac{d\vec{x}}{(2\pi)^d} e^{-g^2G_0}\left(e^{g^2G(x)} - 1\right) e^{i\vec{q}\vec{x}}\int d\vec{p}\,\frac{F(p)}{\sqrt[4]{p^2+1}}\,\frac{M}{\sqrt[4]{\left((\vec{q}-\vec{p})^2+M^2\right)}}$$

The integration over the angles can be performed using these relations:

$$\int_0^{2\pi} d\theta\cdot\frac{1}{(1 - z\cos\theta)^\alpha} = \frac{2\pi}{(1+z)^\alpha}\,{}_2F_1\left(\alpha,\frac{1}{2};1;\frac{2z}{1+z}\right) \text{ та} \tag{66}$$

$$\int_0^{2\pi} d\varphi\cdot e^{iqx\cos\varphi} = 2\pi\cdot J_0(qx). \tag{67}$$

Thus for *I(q)* we have:

$$I(q) = q^2\int \frac{d\vec{x}}{(2\pi)^2} e^{-g^2G_0}\left(e^{g^2G(x)} - 1\right) e^{i\vec{q}\vec{x}} = q^2\,\frac{e^{-g^2G_0}}{(2\pi)^2}\int_0^\infty\int_0^{2\pi}\left(e^{g^2G(x)} - 1\right) e^{iqx\cos\varphi}\,x\,dx\,d\varphi =$$
$$= \frac{e^{-g^2G_0}}{2\pi}\,q^2\int_0^\infty\left(e^{g^2G(x)} - 1\right) J_0(qx)\,x\,dx = \frac{e^{-g^2G_0}}{4\pi}\int_0^\infty\left(e^{g^2G\left(\sqrt{x}/q\right)} - 1\right) J_0\left(\sqrt{x}\right)dx \tag{68}$$

For *P(q)* we have:

$$P(q)=\int d\vec{p}\,\frac{F(p)}{\sqrt[4]{p^2+1}}\frac{M}{\sqrt[4]{\left((\vec{q}-\vec{p})^2+M^2\right)}}=\int_0^\infty\int_0^{2\pi}\frac{F(p)}{\sqrt[4]{\left(p^2+1\right)\left(q^2+p^2-2qp\cos\theta\right)}}pdpd\theta=$$
$$=\int_0^\infty\frac{F(p)}{\sqrt[4]{\left(p^2+1\right)\left((q+p)^2+M^2\right)}}2F_1\left(\frac{1}{4},\frac{1}{2};1;\frac{4qp}{(q+p)^2+M^2}\right)pdp \qquad (69)$$

During specific calculation of this integral, the integral's interval is changed to [0,1] using substitution $p=\frac{1}{z^2}-1$. Then we have for e_2:

$$e_2=f_2\cdot\left(1-e^{-g^2G_0}\right)-J;$$
$$J=\frac{(4\pi)^2}{2}M\int_0^1\frac{zI\left(\frac{1}{z^2}-1\right)P\left(\frac{1}{z^2}-1\right)}{\left(1-z^2\right)^4\sqrt[4]{z^4+\left(\frac{1-z^2}{M}\right)^2}}dz \qquad (70)$$

where the integral is transformed to [0,1] using the above substitution. Normally, calculating e_2 would have required calculation of 3-dimensional integrals. Since the integrand factorizes into $I(q),P(q)$ that can be calculated independently, this calculation significantly simplifies. This is even more important since the function $I(q)$ does not depend on which component of the linear superposition *F(q)* is calculated, and can be evaluated one time for all components of the form-factor.

Regarding numerically calculating *C[F]*, we must also mention that in the terms e_0, e_1 the function *G(x)* enters the integral with a constant. From the analytical form of this function for chosen form-factor one can see that at zero this function is not smooth. In particular, at zero this function behaves as ln(*x*). This creates difficulties during calculating the integral $\int\frac{\left(1-e^{G(x)}\right)}{x^2}e^{-x}dx$ in e_1, which will thus have irregular behavior at small *x* even though the integral is converging. In order to remove these problems, we use explicit expansion of *G(x)* in *x* up to certain order in x^n.

Another a more complicated problem arises form the calculation of the integrals e_2, since these integrands contain an oscillating function $J_0(qx)$ for large q. Again, this creates problems with evaluating such integrals numerically with a required precision. In order to work around this problem for large *q*, the integral $I(q)$ was replaced with its asymptotic expansion in *q* up to fifth order in 1/*q*. The algorithm calculating $I(q)$ adaptively changed the algorithm to calculation using large *q* series expansion when the last term of the series be-

came smaller a certain prior-set threshold. Asymptotic expansion of *I(q)* is very complex, and for this reason is not presented here.

Summarizing, having developed numerical schemes for calculating all terms of the functional, we represent it in the following form:

$$C(\{a,\alpha\}) = \frac{1}{g}\left(f_1(\{a,\alpha\}) - \frac{f_2(\{a,\alpha\}) - e_2(\{a,\alpha\})}{e_0(\{a,\alpha\})} - \frac{f_3(\{a,\alpha\}) + e_1(\{a,\alpha\})}{g e_0(\{a,\alpha\})} \right) \qquad (71)$$

Important advantage of this form and choice of the form-factor in the form (57) is that the dependence of the functional on the coupling constant enters algebraically in the form of factors in *f* and *e*, and not in the exponents in the integrals *e*. This alleviates many difficulties during systematic calculation of the renormalization for different *g*.

Presence in the problem of a large number of special function and 1-dimensional integrals motivates using for the calculation a modern computer mathematical system. In particular, Matematica 3.0 has excellent list of features allowing it to quickly calculate integrals and a large number of special functions. In this work, abilities of this system were taken as the base for the subsequent computational scheme, which was realized in Matematica 3.0. Using standard functions of this system, simple integrals were calculated with precision of not worse than 10^{-8}, and the complex 3-dimensional integral e_2, with account for the comments made above, with precision of not worse than 10^{-6}. This precision is considered by the author as sufficient for the calculation, since it coincides with the best precision offered by the used form-factor (57). Increasing the precision of the integral e_2 results in significant growth of the computational time while not leading to any novel scientific results in this case.

For comparison, let us present here the time required to calculated different parts of the functional *C[F]*. Simple integrals $f_{1\text{-}3}$ required from Matematica 3.0 approximately up to one second, and the most time is spent on f_2. Calculation of 1-dimensional integrals in $e_{0\text{-}1}$ requires approximately 1-2 sec. The most time-consuming calculations are indeed e_2, which takes from 10 to 50 seconds or more.

After realizing the numerical scheme for the functional *C[F]* the problem reduces to finding a minimum in many-dimensional nonlinear variational problem. For solution of this problem in parametric form $\{a,\alpha\}$ we used a custom approach. Generally, solving this problem for one component of the form-factor is trivial, since corresponds to finding a maximum on a plain (a,α), which can be achieved using a standard algorithm. However, already for two components standard algorithms do not give a good result. A very important issue

becomes a good choice of the starting point in these algorithms. In this work, for the solution of the variational problem with two or more parameters we used an approach of prior adaptive stochastic optimization, developed in my Bachelor work (1999), and fine-tuning using a gradient-descant-type method.

Adaptive modification of the stochastic search used in this form consisted of organization of the stochastic search in the form of several steps with reducing search area, where minimum was identified stochastically every time. Respectively, the center of the new search window every time was selected as the current best-found minimum from the prior stage, and the degree to which the search area was reduced was chosen so that the probability of loosing real minimum for certain “good” energy-surface did not exceed a pre-set value.

Used stochastic search procedure had following methods over regular optimization methods:

Not sensitive to the choice of initial poitn;

Can leave one local minimum for a different minimum during search;

Due to adaptive reduction of the search window has good efficiency and precision for the final result.

We should also note that, as the calculations had shown, the accuracy of such prior stochastic approximation already had generally the order of $10^{-3} \div 10^{-4}$, which without doubt is sufficient for constructing any graphs and somewhat exceeds the accuracy provided by two-component form-factor (57). Therefore, for construction of the graphs the final fine-tuning of the optimum was not performed.

In order to control convergence and effectiveness of stochastic optimization we also performed the calculations using certain specific mass and coupling constant ($M = 6.944$ – the relation between the masses of the real nucleons and π-mesons; $g = 1$) up to 4-component form-factor inclusively. This calculation allowed us to evaluate the precision of regular optimization in $10^{-5} \div 10^{-6}$, close to the precision provided by used 4-component. Table 1 gives an overview of these results:

Table 1: M=6.944; g=1

	C	M_0	a/α	a/α	a/α	a/α
	11.971953	14.645327	0.80962 0.79888			
	11.97786	14.648128	0.6359 0.95406	0.2086 0.89722		
	11.978387	14.648378	0.5144 1.0504	0.27364 0.6008	0.09009 1.1239	
	11.978487	14.648425	0.50271 1.10048	0.1899 0.64956	0.04462 1.30257	0.14476 0.59870

In addendums 1-4 we also present the results of systematic calculations of the bare mass M_0 in relation to the physical mass M. The graphs describe the dependency of the mass M_0 on g for M={0.1, 1, 6.944, 10, 100}, calculated using 2-component form-factor, and also the dependence on g of $C[g,M]$.

As can be seen there, for all values of M and g "bare" mass M_0 is a monotonous function in g, smooth, and with a wide minimum at zero, without any nonregularities. Bare mass is always *larger* than the physical mass (the effect of attraction in the model). One can also see that in all graphs the results for N=1 and N=2 components in (57) practically cannot be distinguished. Only for very large mass (M=100) a noticeable difference between these two approximations develops. In Figure 1.4 the same dependencies are shown for different masses. It can be seen that the most significant effect of renormalization is for small mass, where the effect is several times stronger than for large M.

In addendum 2 also presented are the dependencies of $C(g)$ for different M. It can be seen that the renormalization constant is smooth decreasing function with bell-shaped maximum at zero. For large mass M=100 the form-factor (57) turns out to be not sufficiently good, reflected in that fact that one-component fails to describe well the real situation, and calculated function $C(g)$ is not even smooth. At the same time, the difference between N=1 and N=2 in (57) is approximately one, which in comparison with $C \approx 20$ gives relative precision 0.05. In this thesis we also constructed the graphs of the constant of renormalization M_0 / M as a function of mass, and compared that with the approximations given in weak coupling (addendum 3). It can be seen that good agreement is achieved through $g \approx 1 \div 2$, which can be therefore called the limits of the perturbation theory in this case. Furthermore,

similarity of the approximated and exact results can be observed for large masses, in agreement with the approximation of heavy nucleons. Also is seen that the bare mass is always smaller than that given by the weak coupling approximation.

From systematic calculations we can make conclusion that already one-component Yukawa-approximation to the form-factor can give precision sufficient for most practical applications.

In this thesis we also investigate certain simplification of the variational problem. In particular, we investigated if the term e_2 can be neglected, and also if one can vary the radius α. We furthermore will call these approximations as the approximation "e_2=0" and the approximation of the "constant form-factor radius" (CFR), respectively. In the graphs 1-3 of addendum 4 we present the dependencies of the effectiveness of these approximations (in the form $M_{0_{набл.}} / M_{0_{точне}}$). We can see that "$e_2$=0" gives significantly worse result already for $g \approx 1 \div 2$, while CRF remains good for all g especially for intermediate values of g where the accuracy of this approximation is up to 98%.

From the graphs it also can be seen that the effectiveness of the approximation "e_2=0" monotonously drops up to 80% in the case of intermediate masses. The worst result is observed for small masses, where accuracy drops to 50% for $g \approx 5$. Since this approximation is in certain way related to perturbation expansion ($e_2 \approx g^2$), we can make a conclusion that for such methods the important parameter is not only the smallness of the coupling constant but also largeness of the mass of the nucleons relative to that of the exchange particles. It is also interesting to not that the quality of the approximation for large mass M=100, where the quality has a minimum and then increases. From presented graphs we conclude that the approximation "e_2=0" is not sufficient with accuracy in 10-20% for intermediate values of mass.

With regard to CFR approximation we observe that it is effective for all mass ranges (>95%). As with the case "e_2=0", the worst described case is that of light nucleons. Interesting quality of this approach is that the best described are the intermediate masses. For physical mass M=6.944 the precision of this approximation is 98%, which is an excellent result.

6. Two-particle problem

Variational principle allows not only developing nonperturbative schemes for renormalization, but also deriving effective equations for many-particle states, both within the regular perturbation theory (if considered anzats are expansions of states in perturbation series such as $|\Psi_2\rangle = \phi_0|N_1N_2\rangle + \phi_1|N_1N_2\pi\rangle + \phi_2|N_1N_2\pi\pi\rangle$) as well as outside of the perturbations. In the case of arbitrary space dimensionality, nonperturbative relativistic two-particle equation for scalar Yukawa model was described in section 3. It is interesting that dependence on the bare mass in this case can be replaced with physical mass. Furthermore, the form-factor *F(q)* entering this problem should be taken from the solution to 1-state problem.

The relationship (29) is integral equation in many-dimensional space that has complex form and, therefore, has to be solved numerically. In 2D, this equation requires calculating 5-dimensional integrals. Although this is not a fundamental problem, it makes solving equation (29) a very difficult problem. In this work, we only perform an analysis of this equation for small and large g. Our starting point is the relationship (29) where all vector quantities are 2D and all integrals are double integrals.

6.1 Weak coupling

To construct the two-particle equation in the approximation of weak coupling, it is necessary to obtain the quantities $\tilde{H}(p,q;g), \tilde{\Omega}(p;g)$ in $g \to 0$. This can be done relatively easily if we use the results of section 4 and take into account that $I(p) \approx g^2 F^2(p)$. Then, in the leading term of perturbations we obtain:

$$\begin{aligned} \tilde{\Omega}(p) &= \int d\vec{q}\,\Omega(\vec{q}-\vec{p})\left(\delta(\vec{q})e^{-g^2G_0} + I(\vec{q})\right) \approx \omega(p,M) + O\left(g^2\right), \\ \tilde{H}(\vec{p},\vec{q}) &= \int d\vec{q}\cdot H(\vec{q}'-\vec{p},\vec{q})\left(\delta(\vec{q})e^{-g^2G_0} + I(\vec{q})\right) \approx H(-\vec{p},\vec{q}) + O\left(g^2\right) \end{aligned} \qquad (72)$$

from where, taking leading terms from all components of (29) we obtain:

$$(2\omega(p,M)-M_2)\Psi(p)=2g^2\int d\vec{q}\left(\frac{2H(-p,q-p)F(q-p)}{\sqrt{\omega(q-p)}}-\omega(q-p)F^2(q-p)\right)\Psi(q)\approx$$

$$\approx 2g^2\int d\vec{q}\frac{M}{\omega(q-p,M)\left[\omega_{q-p}+\omega(q-p,M)-M\right]^2}\times$$

$$\times\left\{2\sqrt{\frac{M}{\omega(p,M)}}\frac{\omega(q-p,M)}{\omega(q,M)}\frac{\omega_{q-p}+\omega(q-p,M)-M}{\omega_{q-p}}-1\right\}\Psi(q) \qquad (73)$$

As one can see, even in the case of weak coupling the equation is very complex for numerical solution. From mathematical viewpoint this is determined primarily by complexity of the form-factor *F(q)* (47). Another essential feature of this equation is repulsion at large moments $p>>M$ and $q>>M$, which also makes analytical analysis difficult. Subsequent consideration can be performed in separable approximation which essentially takes into account the behavior of the equation at small moments.

According to the general procedure of separable approximation, for the core of the integral equation $V(p,q)$ we use its first term in the separable Beithman approximation: $V(p,q)\approx\frac{V(p,0)\cdot V(0,q)}{V(0,0)}$, after which the equation can be easily solved. Using of this method in our case leads to the relation:

$$1=4\pi\cdot g^2\int_0^\infty\frac{1}{\omega_z{}^2\left(\omega_{Mz}+M(\omega_z-1)\right)^4}\left\{2\frac{\omega_{Mz}+M(\omega_z-1)}{\omega_{Mz}}-1\right\}\times$$

$$\times\left\{2\frac{\omega_{Mz}+M(\omega_z-1)}{\omega_{Mz}}-1\right\}\frac{Mz}{2\left(\omega_z-\frac{M_2}{2M}\right)}dz \quad , \qquad (74)$$

behavior of this equation when the coupling energy $2M-M_2\to 0$ can be found easily and is logarithmic. This means that for $g\to 0$ the equation has the form as in:

$1=4\pi\cdot g^2\left(C_1\ln\left(\frac{2M-M_2}{2M}\right)+C_2+\ldots\right)$, and the coupling energy behaves as

$$\varepsilon=\frac{2M-M_2}{2M}\approx C\cdot e^{-\frac{1}{g^2}},\ g\to 0\,. \qquad (75)$$

As we can see, the energy of the bound state depends on *g* in nonanalytic way, which is also encountered in the theory of superconductivity and some other problems. Integral (74) can be calculated exactly in the limit of small masses, which gives an equation in explicit form:

$$1=-2\pi\cdot g^2M\frac{(\ln\varepsilon+1-\varepsilon)}{(1-\varepsilon)^2}.$$

In relation to the case of weak coupling, interestingly that corresponding equation in 3D prohibits existence of solutions for very small *g*, since the integral in (74) is bounded when $M_2 \le 2M$.

Completing the consideration of the weak coupling case, let us determine the situation when the mass of the nucleons is much larger than that of the mesons. It is easy to perform respective expansion in (73) and obtain for the core of the integral equation:

$$V(p,q) \approx \frac{1}{\omega_{q-p}{}^2} - \frac{1}{M^2}\left(\frac{|q-p|^2\left(2+3|q-p|^2\right)}{4\omega_{p-q}{}^4} + \frac{\vec{p}\vec{q}}{\omega_{p-q}{}^2} \right) + \ldots;$$
$$(2\omega(p,M) - M_2)\Psi(p) \approx \left(\frac{p^2}{M} + 2M \cdot \varepsilon \right)\Psi(p) = 2g^2 \int d\vec{q} \cdot V(p,q) \qquad (76)$$

This is the regular Schrodinger equation with respective Yukawa potential given by the first term in (76). The second term describes effective repulsion, which is the reflection of repulsion for large moments.

6.2 Strong coupling

As we mentioned in section 4, essential quality of variational method is the possibility of formulating the results in the form that is correct both for small and strong couplings. This allows one to consider both cases within the same mathematical framework. In our case, this allows to relatively simply consider the case of large *g* in the problem of mass renormalization, that in turn opens the way for similar calculation in two-particle states.

In order to solve this problem, we will consider that for large couplings the system collapses in space representation. Respectively, in momentum representation the wave function approaches a constant (the Fourier image of δ-function). Therefore, we will attempt to solve equation (29) departing from these assumptions. Also, we will need the results from the section 4.1 establishing the important properties of the form-factor *F(q)* for large *g*. In particular, we need the expression for *I(q)* when $g \to \infty$. According to (49,50), this relation has form:

$$I(q) = \frac{1}{\pi^{d/2}} \frac{1}{\bar{g}^d} \exp\left(-\frac{q^2}{\bar{g}^2} \right), \text{ where} \qquad (77)$$
$$\bar{g} = 2gA \approx 2A^* \sqrt{Mg} \approx \sqrt{g}$$

Let us analyze now the behavior of the kinetic and potential energies for large couplings. For that we consider the expressions $\tilde{F}(p)=\int d\vec{q}F(q)I(q+p), g\to\infty$, where F can be some function representing the kinetic or potential energy:

$$\tilde{F}(p)=\int d\vec{q}F(q)I(q+p)=\int\frac{d\vec{q}}{\pi^{d/2}}F(q)\frac{1}{\bar{g}^d}e^{-\frac{(\vec{q}+\vec{p})^2}{\bar{g}^2}}=$$

$$=\int\frac{d\vec{Q}'}{\pi^{d/2}\bar{g}^d}F\left(\vec{Q}'-\vec{p}\right)\cdot e^{-\frac{\vec{Q}'^2}{\bar{g}^2}}=\int\frac{d\vec{Q}}{\pi^{d/2}}F\left(\bar{g}\vec{Q}-\vec{p}\right)\cdot e^{-Q^2}\approx \quad (78)$$

$$\approx\int\frac{d\vec{Q}}{\pi^{d/2}}\bar{g}^{\alpha}f\left(\vec{Q}-\frac{1}{\bar{g}}\vec{p}\right)\cdot e^{-Q^2}\approx\bar{g}^{\alpha}\int\frac{d\vec{Q}}{\pi^{d/2}}f\left(\vec{Q}\right)\cdot e^{-Q^2}+\ldots$$

where we make approximation $F(a\cdot\vec{q})\approx a^{\alpha}\cdot f(\vec{q}), a\to\infty$, and otherwise the integrals are expanded in $\frac{1}{\bar{g}}=\frac{1}{\sqrt{g}}\to 0, g\to\infty$. Taking into account (78), for the potential energy we obtain:

$$\tilde{H}(p,q-p)\approx\int\frac{d\vec{q}'}{\pi^{d/2}}\frac{M}{\sqrt{\omega(q',M)\omega(\vec{q}'+(\vec{p}-\vec{q}),M)}}\frac{e^{-\frac{(\vec{p}+\vec{q}')^2}{\bar{g}^2}}}{\bar{g}^d}\approx$$

$$\approx\int\frac{d\vec{q}'}{\pi^{d/2}}\frac{M}{\sqrt{\omega(\bar{g}q',M)\omega(\bar{g}\vec{q}'+(\vec{p}-\vec{q}),M)}}e^{-\left(\vec{q}'+\frac{1}{\bar{g}}\vec{p}\right)}\approx \quad (79)$$

$$\approx\frac{M}{\bar{g}}\int\frac{d\vec{q}'}{\pi^{d/2}}\frac{1}{q'}e^{-(\vec{q}'+1/\bar{g}\cdot\vec{p})^2}\approx e_0\left(1+O\left(\bar{g}^{-1}p\right)\right), |\vec{q}-\vec{p}|<<\bar{g}$$

,

where the coefficient α equals negative one, and where we take into account equation (42). In order to investigate existence of the bound state in this problem we consider the potential:

$$V(p,q-p)=-\left(2\frac{F(q-p)}{\sqrt{\omega_{q-p}}}\tilde{H}(p,q-p)-\omega_{q-p}F^2(q-p)\right). \quad (80)$$

in the approximation of strong coupling. We take into account that the function $\tilde{H}(p,q-p)$ enters this together with a quickly dropping factor

$$\frac{F(q-p)}{\sqrt{\omega_{q-p}}}\approx\frac{1}{|q-p|^3},$$

therefore, the most important its behavior is that at small values of transferred momentum $|\vec{Q}|=|\vec{q}-\vec{p}|<<\bar{g}$, which is determined by equation (79). Thus, the potential (80) can be changed as:

$$V(q-p)=-\left(2\frac{F(q-p)}{\sqrt{\omega_{q-p}}}e_0\left(1+O\left(\bar{g}^{-1}\vec{p},\bar{g}^{-1}(\vec{q}-\vec{p})\right)\right)-\omega_{q-p}F^2(q-p)\right). \qquad (81)$$

This function in space representation describes a local potential with a minimum at point x=0. Since with growing g the wave function contracts to the region of the minimum potential (x=0), the most important question is the determination of the potential for respective small values of the coordinate x. As can be seen from (32) and (42):

$$V(x=0)\approx-\int d\vec{Q}\left(e_0 2\frac{F(Q)}{\sqrt{\omega_Q}}-\omega_Q F^2(Q)\right)=-e_0 f_1+f_3=-e_0 C(g;M)\approx -C/g<0\,.$$

Thus, we see that at zero the effective potential is infinite and negative. Thus, when $g\to\infty$, the wave function in coordinate representation contracts into a point corresponding to $\Psi^2(x)\to\delta(x)$. By performing calculation in momentum representation with the wave function in the form:

$$\Psi(p)=a^{-d}\varphi(p/a), a(g)\to\infty(g\to\infty), \qquad (82)$$

after averaging equation (29) we obtain:

$$\begin{aligned}&\langle\psi|K|\psi\rangle+E\langle\psi|\psi\rangle=-2g^2\int dx V(x)|\psi(x)|^2\\&E\approx\frac{\langle\psi|K|\psi\rangle-2gC\int dx|\psi(x)|^2}{\langle\psi|\psi\rangle}>0\end{aligned} \qquad (83)$$

As we can see, there is a wide class of wave function for which the energy is negative, which shows that the bound state does exist. It is also possible to show in (82) choosing $a(g)\approx g$ that the coupling constant can be excluded from the equation and the first approximation of the energy E dependency on g is $E\approx g$. Performed calculation shows that, when g grows without bounds the bound state exists in the system, its wave function behaves as $\Psi(p)\approx\varphi\left(\frac{p}{g}\right)$ (or in coordinate representation $\Psi^2(x)\to\delta(x)$), and the bound energy behaves as $M_2\approx -E\approx -g\varepsilon$. (84)

Performed in this section analysis gives the results for two-particle state similar to that performed in [9] but for two-dimensional space. In particular, we obtained answers to the questions about the behavior of the system in the approximations of weak (75) and strong (84) coupling. We should also note that, other than in separable approximation, we did not make any assumptions regarding the dimensionality of space; thus all results are formally correct in spaces of any dimensionality. Of course, since all terms of representation

(32) in this case contain divergent integrals, such results should be only interpreted together with proper renormalization of the theory.

7. Conclusions

In this thesis we consider the problem of description of many-particle states of strongly coupled quantum field theories using variational principle in Hamilton formalism of quantum field theory. We apply this approach for constructing nonperturbative equations for one and two-particle states in Yukawa model, and construct the mass renormalization equations and the effective relativistic equation for two-particle state. In this course, the solution of the one-particle problem turns out to be required for two-particle problem, since it defines the parameters (physical mass M and meson form-factor $F(q)$) that enter the many-particle equations.

The problem of renormalization in this thesis is solved for all values of parameters of the original theory in two-dimensions. This allows obtaining results in analytical form in weak and strong coupling approximation, and heavy and light nucleons. For intermediate values of the coupling constant g and physical mass M a scheme for numerical calculation of renormalization effects with controlled precision is developed, and the calculations are performed for many values of g and M. In this scheme, both the bare mass M_0 and the form-factor $F(q)$ are obtained.

Thus obtained results are used for analytical study of the binding energy of two-nucleon states for small and large coupling constants g.

Performed analytical and numerical calculations show that, in the framework of the above approach, the problem of nonperturbative description of single and two-particle states in scalar model of nuclear interactions can be solved completely in analytically or numerically for all values of the theory parameters with any target precision.

At the same time, developing of the performed work to the case of real 3-dimensional space first needs to realize the program of respective renormalization of encountered divergent quantities. Furthermore, an important unsolved problem is weak suitability of existing integration methods for high-precision evaluation of complex many-dimensional integrals. For example, typically used in such settings method of Monte-Carlo does not allow producing the complex integrals' values with high and controlled precision in reasonable time, and already the 3-dimensional case requires, in the framework of the above approach, numerical high-precision calculations of at least 4- and 5-dimensional integrals.

Therefore, despite the success of the above approach in the case of D=1 and D=2 dimensions, the generalization to D=3 presents an array of highly nontrivial difficulties that can be a subject of future research work.

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8. Literature

[1] Barnes T. and Ghandour G. I. // Ibid.-33, 2305 (1986).
[2] Bethe H. A., Salpeter E. E. Quantum Mechanics of One and Two-Electron Atoms.-Berlin: Springer-Verlag, 1957.
[3] Coleman S. // Phys. Rev.D.-11, 2088 (1975).
[4] Dancoff S. M. // Phys. Rev.-78, 382 (1950).
[5] Darewych J. W. // Ukr. J. Phys.-41,1.-41-50 (1996).
[6] Darewych J. W., Horbatch M. // J.Phys. B.-22, 973 (1989).
[7] Darewych J. W., Horbatsch M., Koniuk R. // Phys. Rev. D.-33, 2316 (1986).
[8] Darewych J. W., Horbatsch M., Koniuk R. // Phys. Rev. Lett.-54, 2188 (1985).
[9] Darewych J. W., Shapoval D. V., Simenog I. V. and Sitenko A. G. // J. Math. Phys.-38, 8.-3908-3924 (1997).
[10] Darewych J. W., Sitenko A. G., Simenog I. V., Sitnichenko A. I. // Phys. Rev.-147, 5.-1885-1897 (1993).
[11] Di Leo L., Darewych J. W. // Can. J. Phys.-70, 412 (1992).
[12] Di Leo L., Darewych J. W. // Can. J. Phys.-71, 365 (1993).
[13] Dykshoorn W., Koniuk R. // Phys. Rev. A.-41,64 (1990).
[14] Ferrell R. A. // Phys. Rev.-84, 858 (1951); Stroscio M. A. // Phys. Rev.-22, 215 (1975).
[15] Grant I. P., McKenzie B.J., Norrington P.H. // Comput. Phys. Communs.-21, 202 (1980); -21, 233 (1980).
[16] Gross F. Relativistic Quantum Mechanics and Field Theory.-New York: J.Willey and Sons, 1993.
[17] Itzykson C., Zuber J.-B. Quantum Field Theory.-New York: McGraw-Hill Books Co., 1980.
[18] Koniuk R., Darewych J. W.// Phys. Lett. B.-176, 195 (1986).
[19] Relativistic, QED and Weak Interaction in Atoms / Ed. By W.Johnson, P.Mohrand J. Sucher.-New York: American Institute of Physics, 1989.
[20] Schiff L. I. // Phys. Rev.-130, 458 (1963).
[21] Simenog I. V., Kuzmenko I. V., Darewych J. W. // Ukr. J. Phys.-40,1-2.-21-28 (1995).
[22] Simenog I. V., Sitenko A. G., Khryapa V. M. and Darewych J. // Ukr. J. Phys.-43, 11.-1346-1355 (1998).
[23] Stevenson P. M. // Phys. Rev.D.-30, 1712 (1984).
[24] Stevenson P. M.// Phys. Rev. D.-32, 1389 (1985).
[25] Tamm I. // J. Phys. USSR.-9, 449 (1945).
[26] Wilson G. K., Walhout T. S. // Phys. Rev. D.-49, 6720 (1994).
[27] Zhang T., Koniuk R. // Phys. Lett. B.-261, 311 (1991).
[28] Бейтмен Г., Эрдейи А. Высшие трансцендентные функции / М.: Наука, 1973.-т.1.-294с.
[29] Боголюбов Н. Н., Парасюк О. С. // Acta Math.-97, 227 (1957).
[30] Боголюбов Н. Н., Парасюк О. С. // ДАН СССР.-100, 25.-429 (1955).
[31] Боголюбов Н. Н., Ширков Д. В. Введение в теорию квантованных полей / М.: Наука, 1984.-600с.
[32] Боголюбов Н. Н., Ширков Д. В. Квантовые поля / М.: Наука, 1980.-320с.
[33] Ефимов Г. В. Нелокальные взаимодействия квантованных полей / М.: Наука, 1977.-366с.
[34] Колинз Дж. Перенормировка: введение в теорию перенормировок, ренормализационной группы и операторных разложений / М.: Мир, 1988.-448с.
[35] Люха В., Шеберл Ф. Сильное взаимодействие / Академ. Экспресс..-Львов, 1996.-182с.
[36] Хепп К. Теория перенормировок / М.: Мир, 1974.
[37] Швебер С. Введение в релятивистскую квантовую теорию поля / М.: Иноиздат, 1963.-842с.

Addendum 1

Renormalization of mass $M_0(g,M)/M$ in two-dimensional scalar Yukawa model

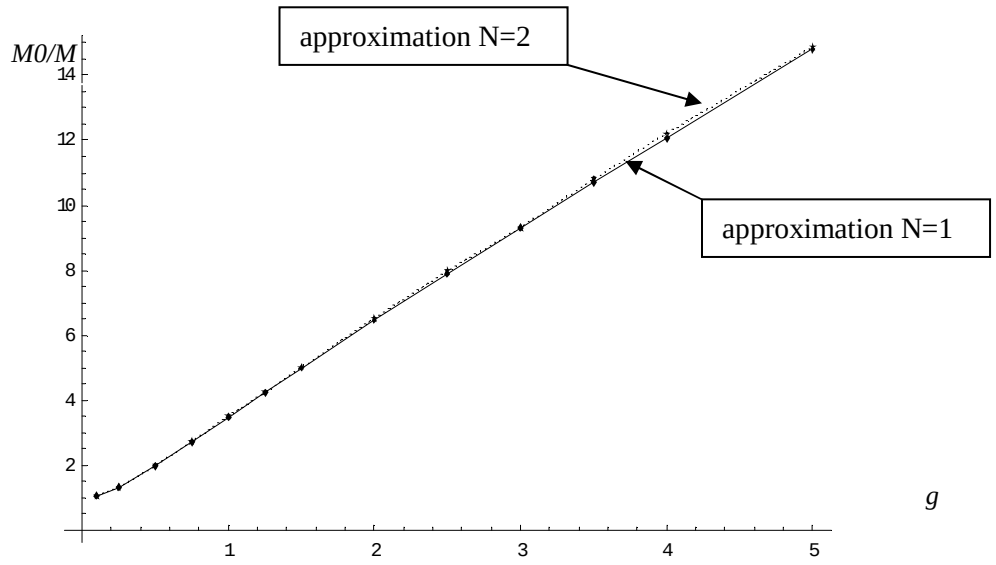

Figure 1 Physical mass M=0.1, approximations N=1,2

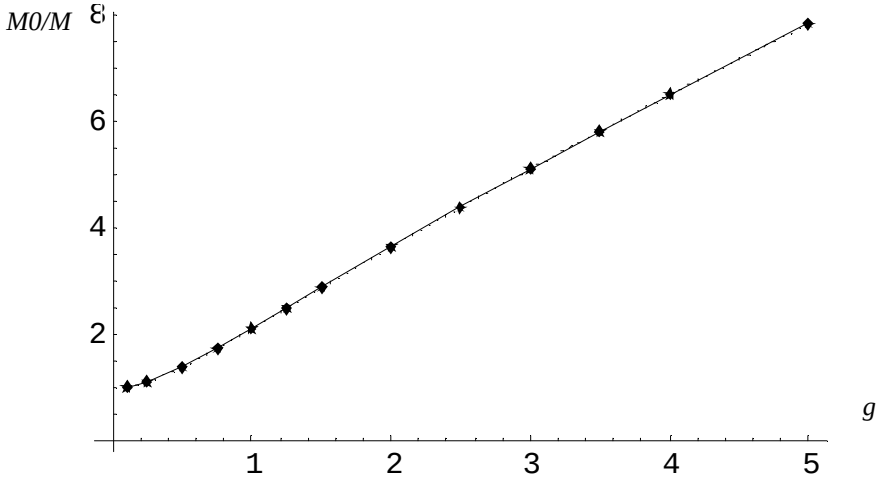

Figure 2 Physical mass M=6.944, approximations N=1,2

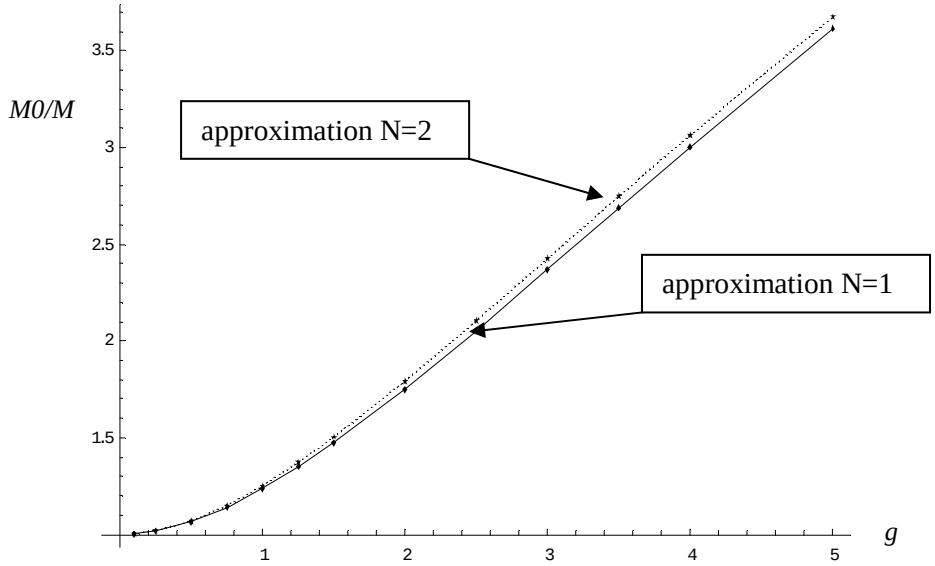

Figure 3 Physical mass M=100, approximations N=1,2

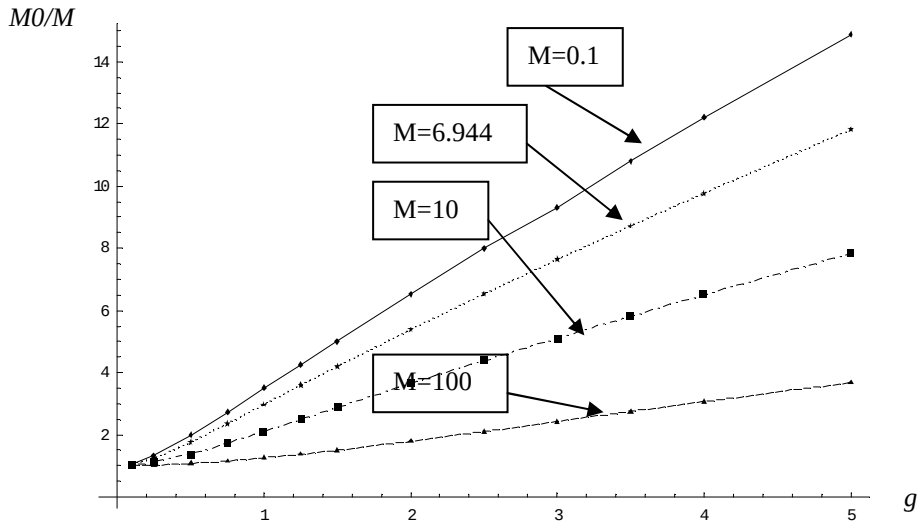

Figure 4 M0/M(g) for M=0.1;6.944;10;100

Addendum 2

The renormalization constant *C[g]* in two-dimensional Yukawa model

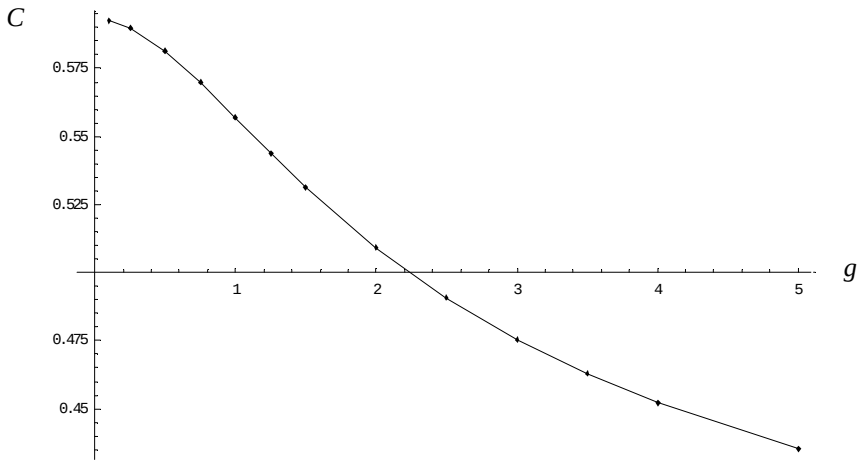

Figure 5 Physical mass M=0.1

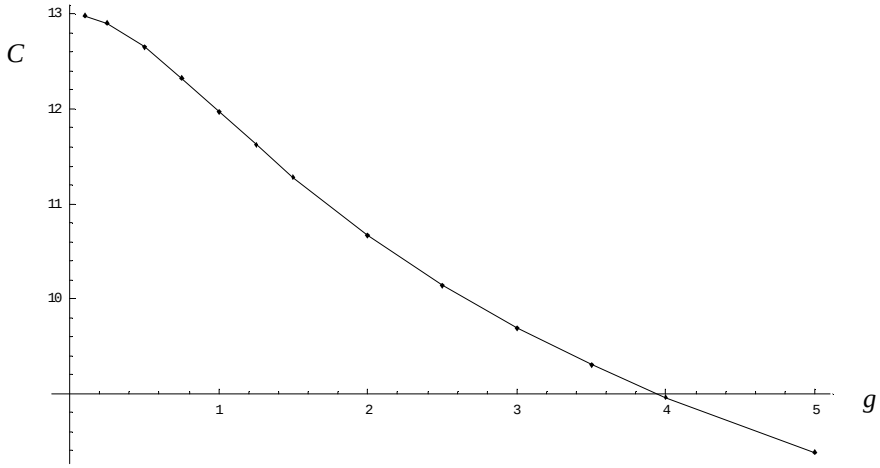

Figure 6 Physical mass M=6.944

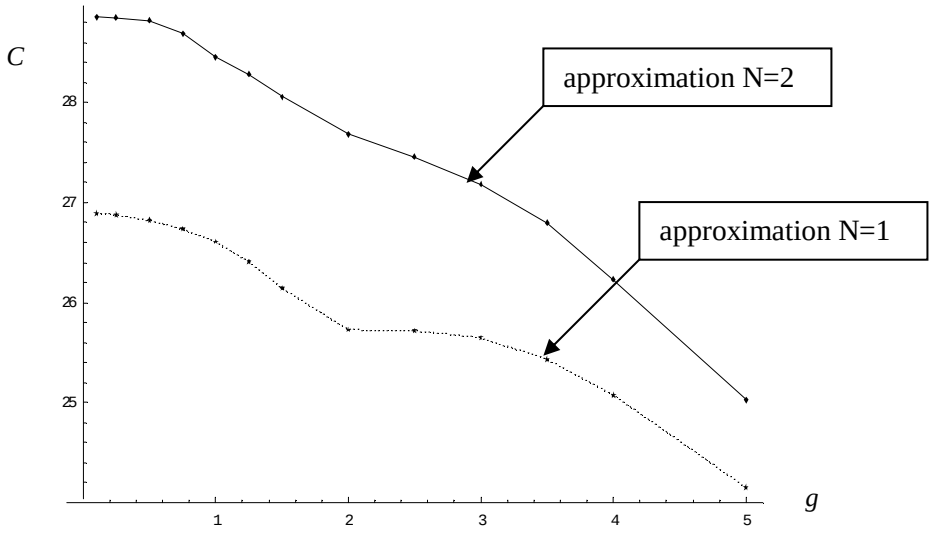

Figure 7 Physical mass M=100

Addendum 3

Dependence of M0/M from physical mass for some coupling constant values

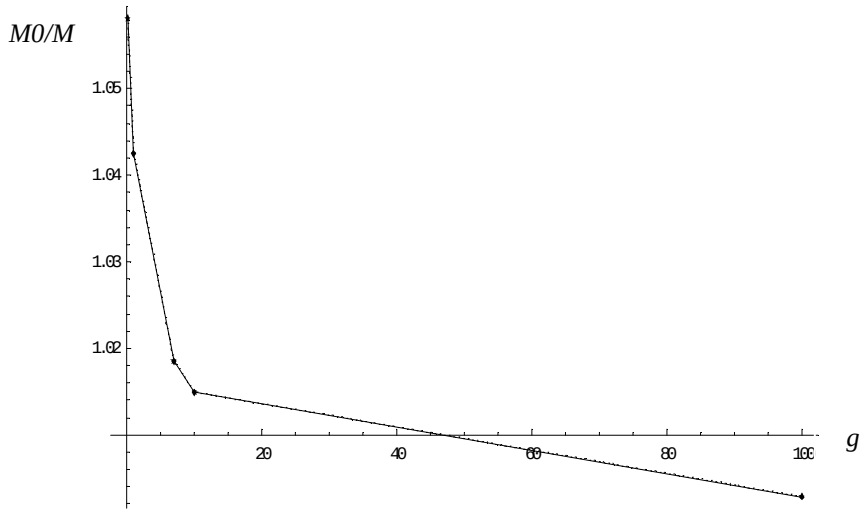

Figure 8 Coupling constant g=0.1

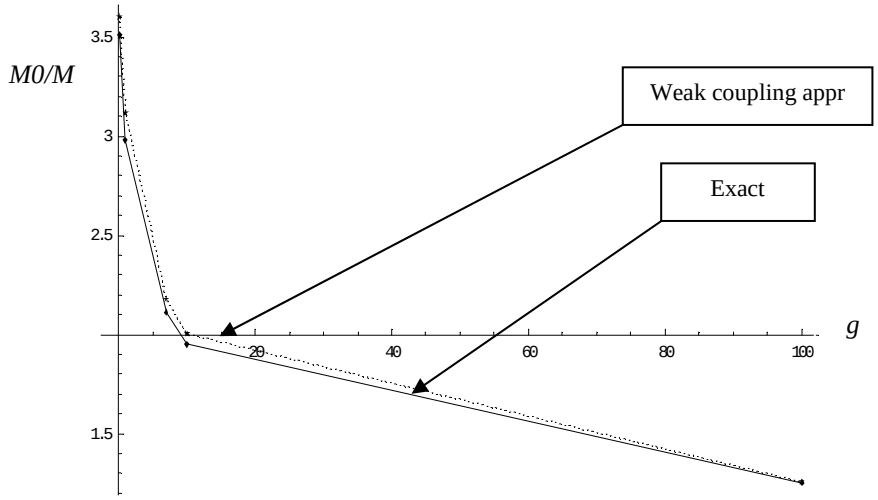

Figure 9 Coupling constant g=1

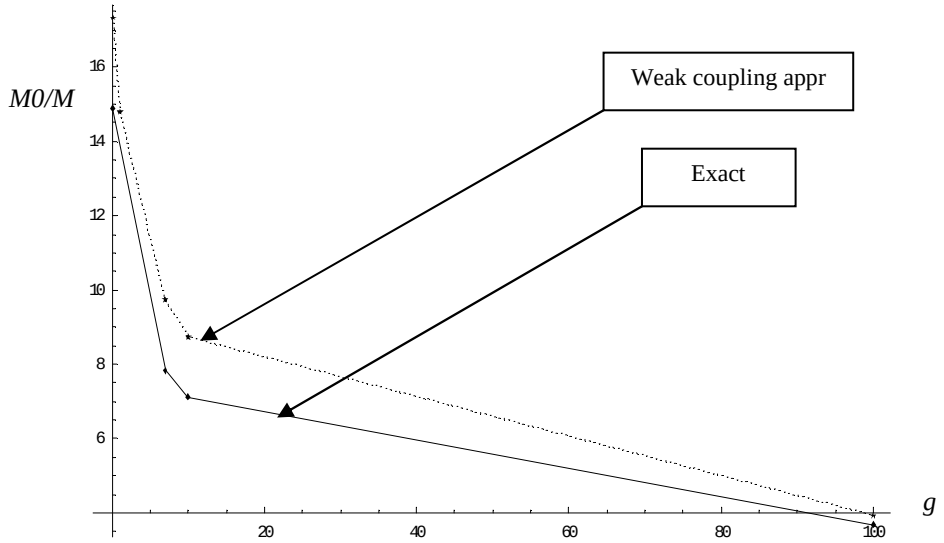

Figure 10 Coupling constant g=5

Addendum 4

Investigation of e_2=0 approximation and Constant Form-factor Radius approximation (CFR)

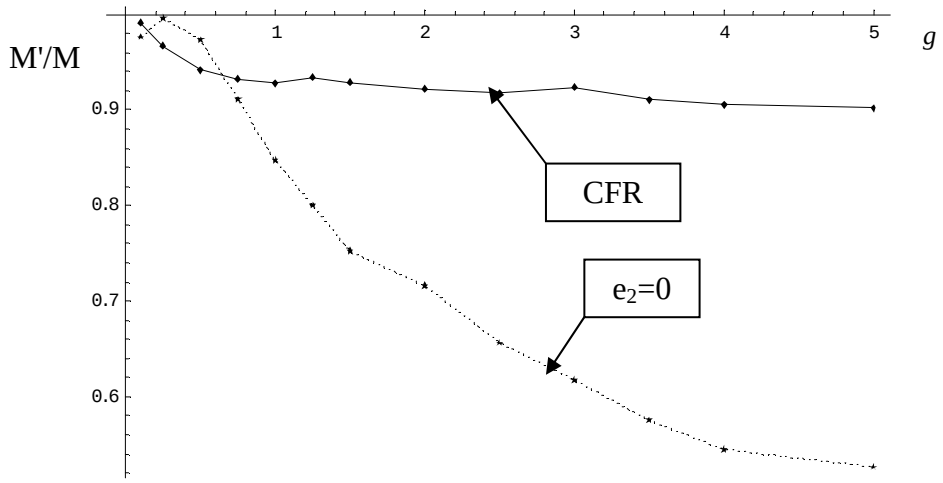

Figure 11 Physical mass M=0.1

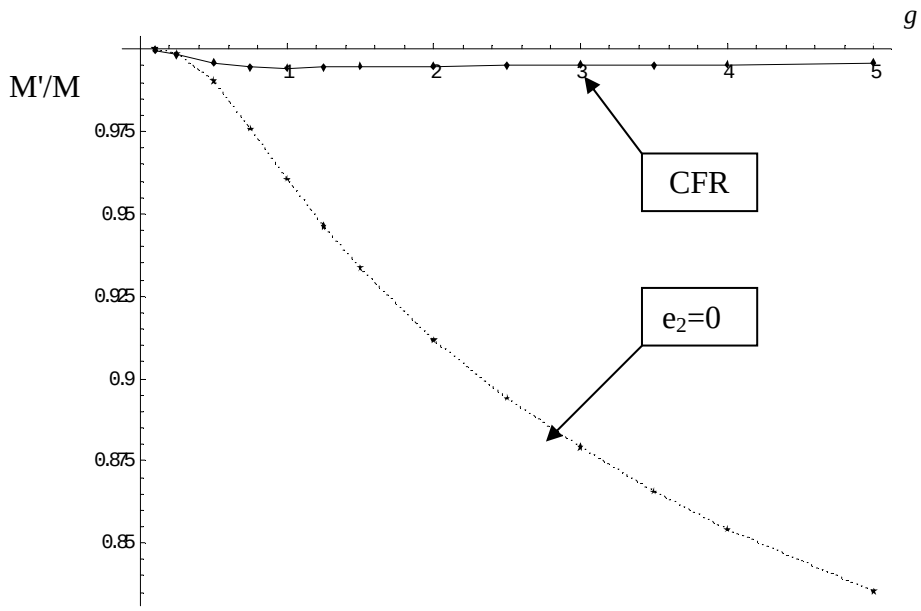

Figure 12 Physical mass M=6.944

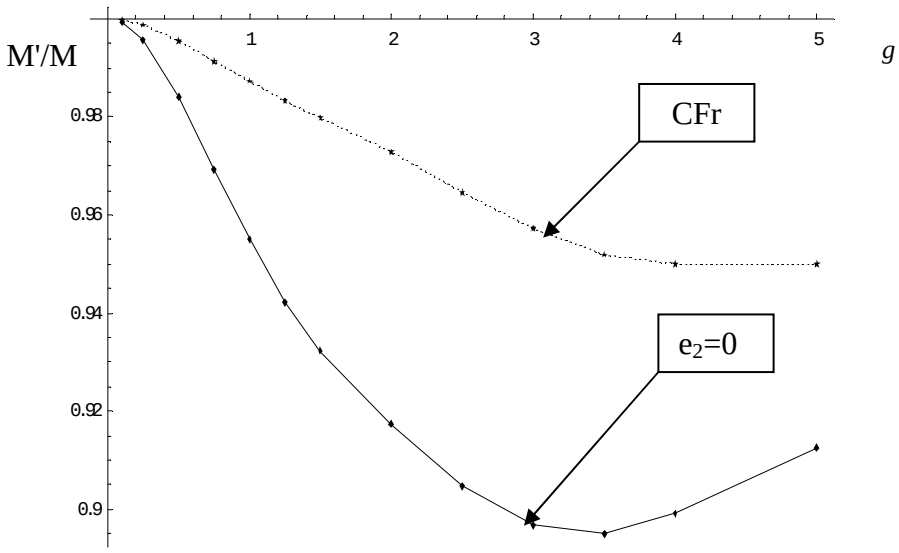

Figure 13 Physical mass M=100